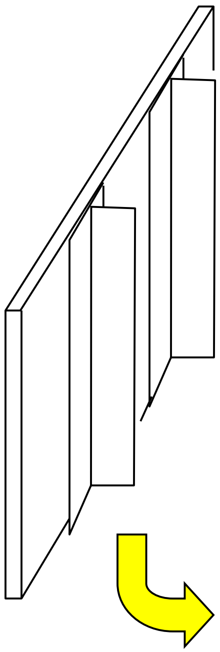
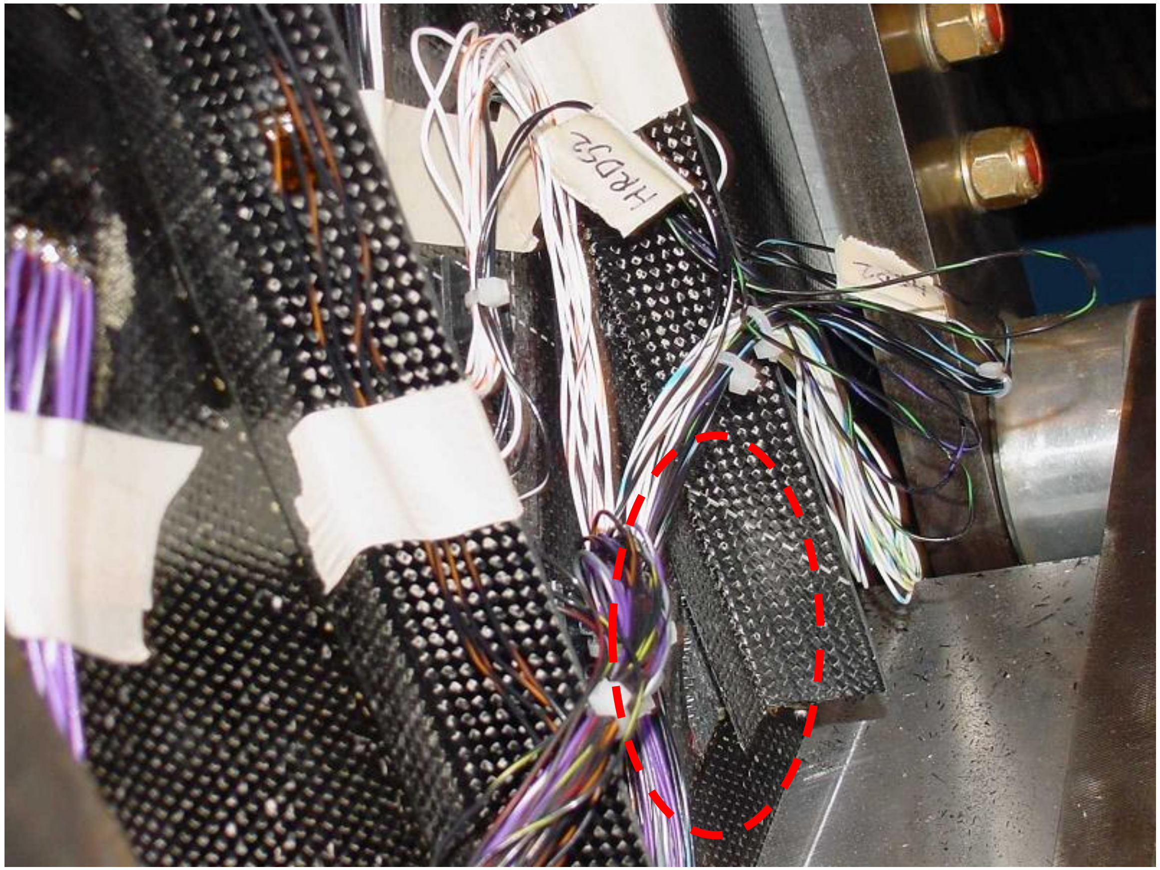
Skin-stiffener separation



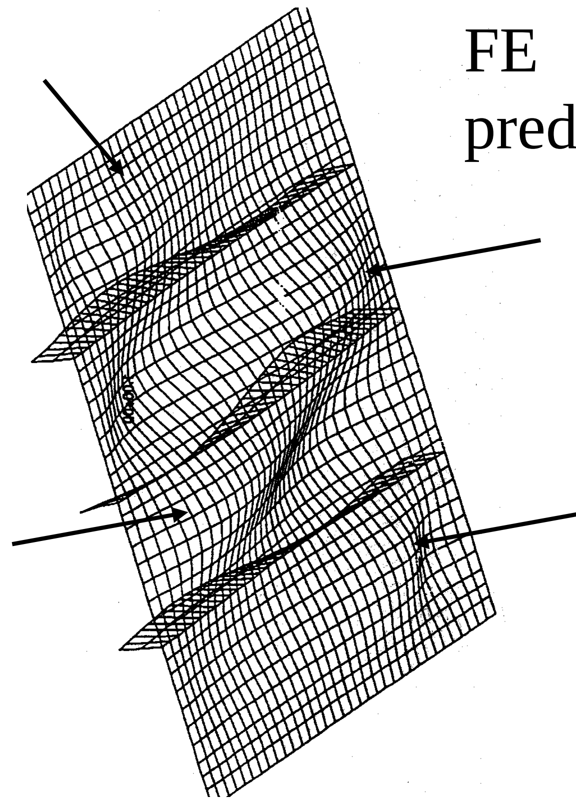


• once the skin buckles, there is a tendency for the stiffener to pull-off from the skin

Skin-Stiffener separation

shear loading

Photo at
400 lb/in

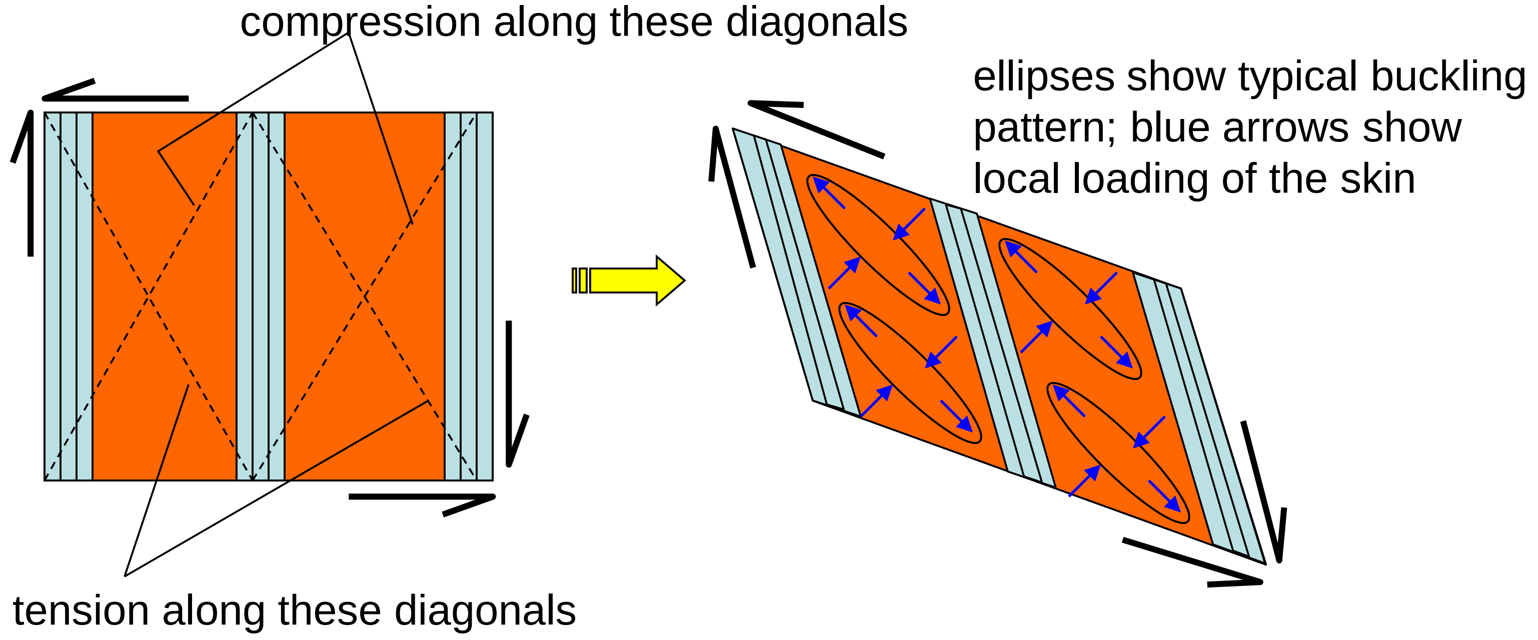


FE
prediction

• arrows point to areas where skin deflects away from the stiffener flange and thus has a tendency to peel off

Skin-stiffener separation

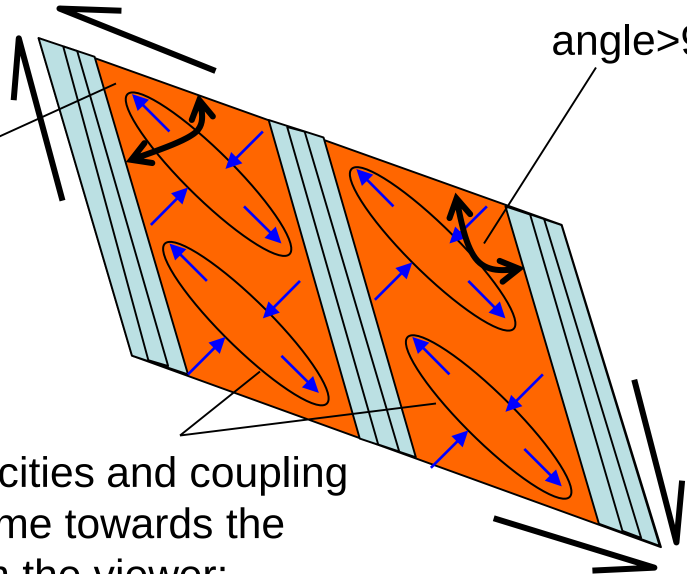
- of particular importance under shear loading is the so-called “pinching” of the skin that can lead to skin-stiffener separation at the corners of the skin bays (between stiffeners)



Skin pinching under shear

angle>90 degrees; skin in tension

angle<90 degrees; skin in compression (pinches)

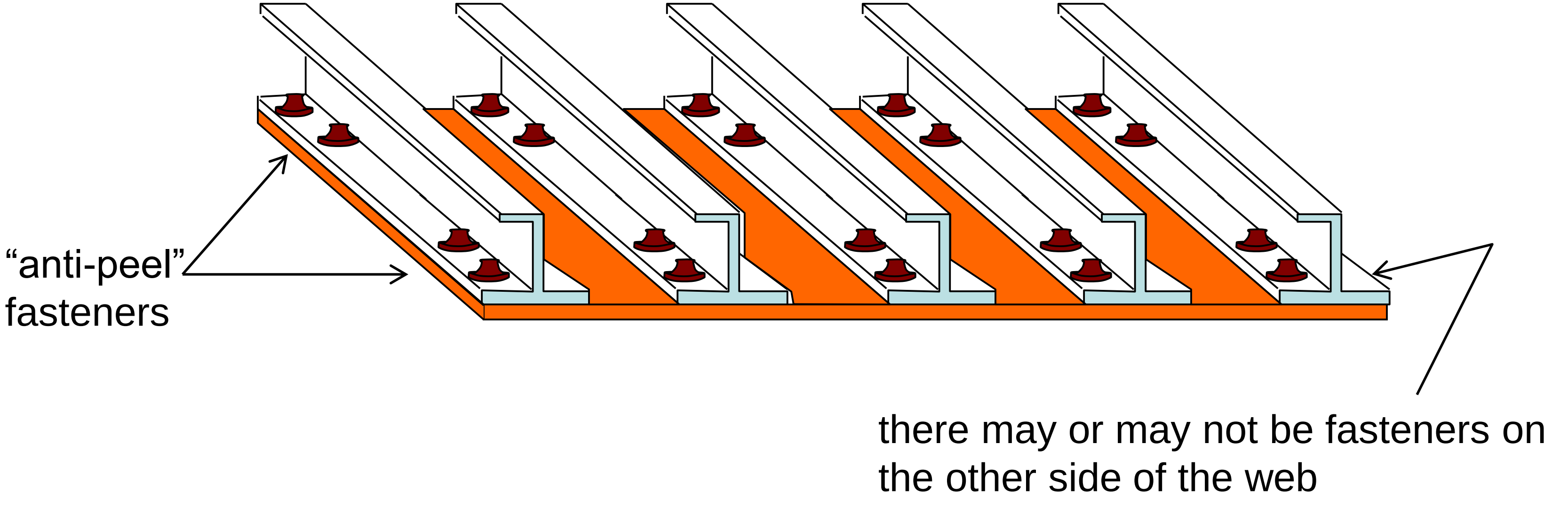


depending on local eccentricities and coupling effects some half-waves come towards the viewer and some away from the viewer; typically, they alternate; thus there will be corners where the skin tends to move away (separate) from the flanges

- typically, the pinched corner (under compression) fails first; this is more pronounced when the skin locally has buckled away from the viewer tending to separate from the stiffener

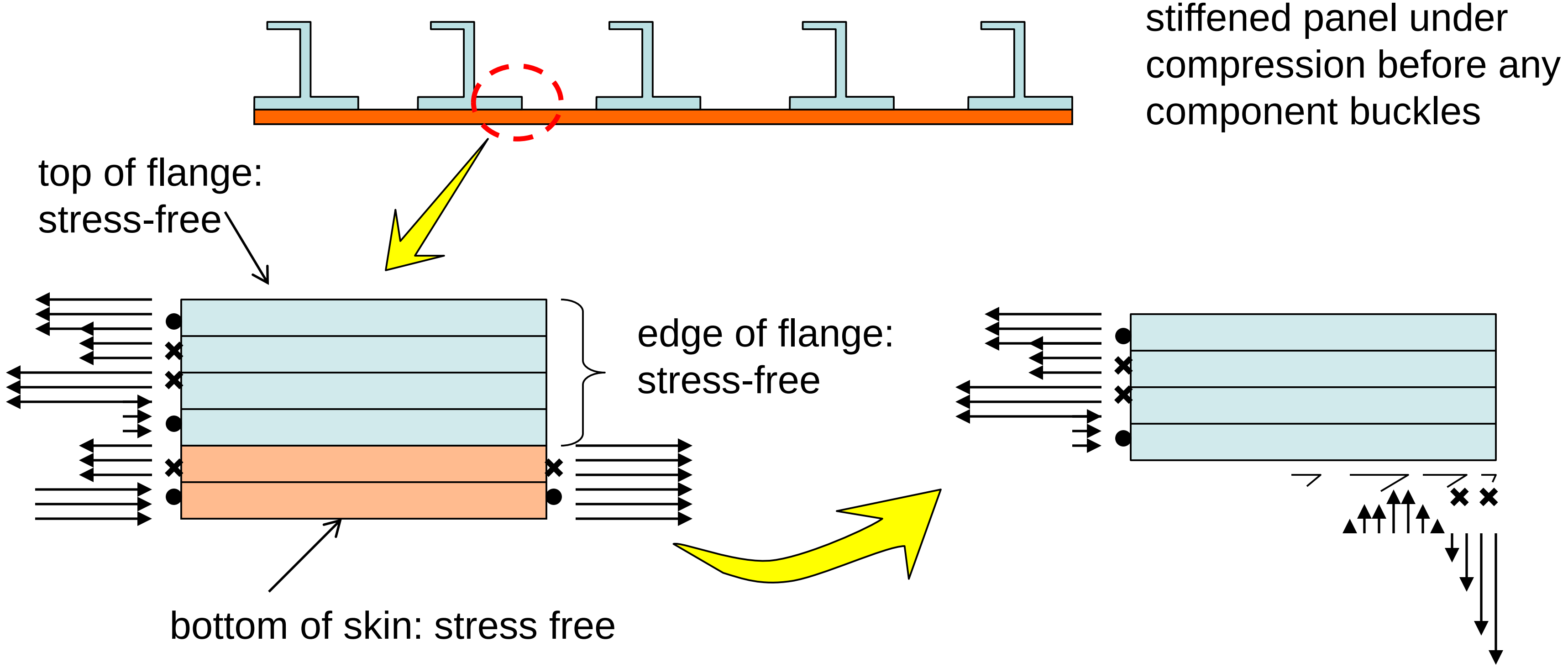
Skin-stiffener separation at bay corners

- since the critical regions (due to pinching and skin-stiffener separation tendency) are at corners of bays, one way to delay at least the separation is to add fasteners at the ends of the stiffeners only to save cost and weight; thus one does not rely on the resin only which is the weakest link between stiffener and skin



Skin-stiffener separation

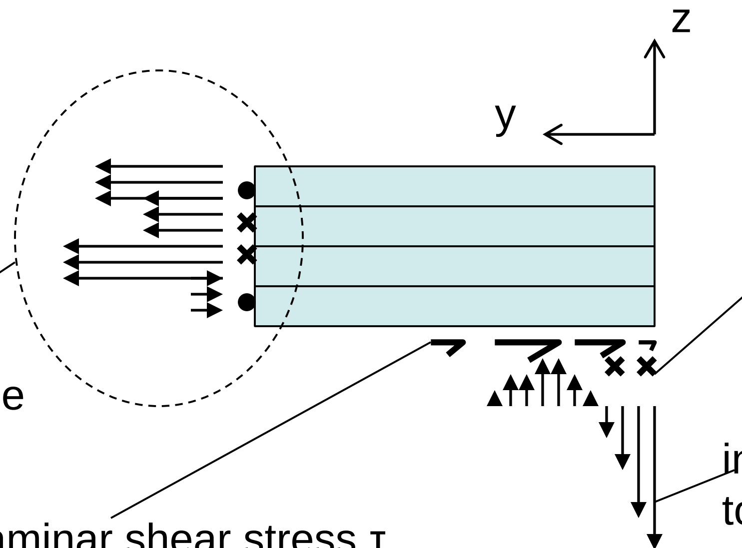
• but even before buckling the tendency for skin-stiffener separation is still there



• interlaminar stresses must develop at the flange/skin interface (and other ply interfaces) to balance the far-field loads

Skin-stiffener separation

free-body diagram of the flange



Classical Laminated-Plate Theory solution rising from compressive load along the axis of the stiffeners

interlaminar shear stress τ_{xz} required to balance forces in the x direction and moments about the y and z axis

interlaminar normal stress σ_z to balance moments about the x axis; from $\Sigma F_z=0$, it must be self-equilibrating

interlaminar shear stress τ_{yz} required to balance forces in the y direction and moments about the x axis

- the interlaminar stresses may combine to cause delamination and thus lead to skin/stiffener separation

Calculation of skin-stiffener separation load(s)

• there are several ways to calculate when the flange may separate from the skin:

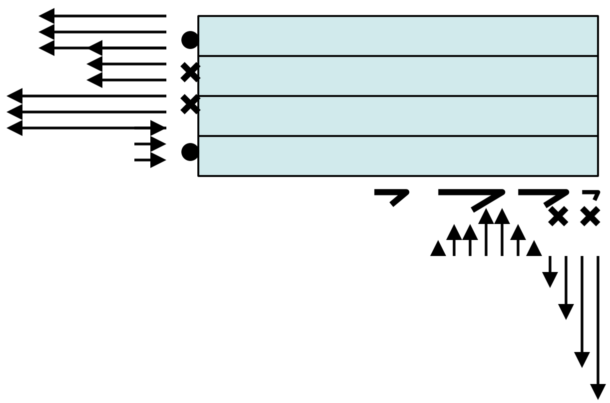
- determine the full 3-D state of stress at the skin/stiffener interface and apply some stress-based delamination criterion

- assume a pre-existing delamination, calculate the energy release rate and determine when it equals the critical energy release rate for delamination propagation

coming up

second part of the course when we talk about delamination

Calculation of interlaminar stresses in skin/stiffener configurations



• to do this we need an additional tool: the Euler-Lagrange equation obtained using calculus of variations

Euler-Lagrange equation: Introduction to calculus of variations[1]

- let I be defined as

$$I = \int_a^b H\left(f(y), \frac{df(y)}{dy}, y \right) dy \qquad (5.4.3.1)$$

with f(a) and f(b) prescribed

- and attempt to find what condition f(y) must fulfill for I to be stationary

- Motivation for doing this: set up the energy in the structure in a form similar to I and minimize it

(1) See for example, Hildebrandt, F.B. *Advanced Calculus for Applications*, Prentice Hall, Englewood Cliffs, NJ, 1976, section 7.8

Euler-Lagrange equation: Calculus of variations

- assume that up to second order derivatives of H with respect to f, df/dy, and y exist and are continuous in the range (a,b)

- the problem becomes: of all **admissible** functions which are the functions that have continuous second order derivatives with the prescribed end values, find the one that makes I stationary

- assume the sought-for function is f(y) and define a family of admissible functions

$$f(y) + \varepsilon v(y)$$

- where ε is a parameter that is constant for each choice of v(y) but may vary for different v(y) functions

Euler-Lagrange equation

• and v(y) is a function that is zero at the end-points a and b and possesses up to at least second order continuous derivatives in the range (a,b)

• thus, f(y)+εv(y) is still an admissible function

• εv(y) is called **a variation** of f(y)

• replace now f in (5.4.3.1) by f+εv and obtain:

$$I(\varepsilon) = \int_a^b H\left(f + \varepsilon v,\, f' + \varepsilon v',\, y\right) dy$$

• since f is the function that makes I stationary, it can be seen that I(ε) is stationary when ε=0

Euler-Lagrange equation

- at the same time, for I to be stationary, must have,

$$\frac{dI(\varepsilon)}{d\varepsilon} = 0$$

- which leads to

$$\int_a^b \left(\frac{\partial H}{\partial (f + \varepsilon v)} v + \frac{\partial H}{\partial (f' + \varepsilon v')} v' \right) dy = 0$$

- since at the same time, ε must be zero,

$$\int_a^b \left(\frac{\partial H}{\partial f} v + \frac{\partial H}{\partial f'} v' \right) dy = 0$$

Euler-Lagrange equation

- use integration by parts to evaluate the second term of the integrand:

$$\int_a^b \frac{\partial H}{\partial f'} v' \, dy = \left[\frac{\partial H}{\partial f'} v \right]_a^b - \int_a^b v \frac{d}{dy}\left(\frac{\partial H}{\partial (f')} \right) dy$$

=0 because v(a)=v(b)=0

- therefore, the condition for I to be stationary when ε=0 is,

$$\int_a^b \left(\frac{\partial H}{\partial f} - \frac{d}{dy}\left(\frac{\partial H}{\partial f'} \right) \right) v \, dy = 0$$

twice differentiable and zero at a and b

- since this eqn must be true for **any** acceptable v(y),

$$\frac{\partial H}{\partial f} - \frac{d}{dy}\left(\frac{\partial H}{\partial f'} \right) = 0 \qquad (5.4.3.10)$$

Euler-Lagrange equation (alternate approach)

• assume that I is allowed to vary and take any of the possible forms in the vicinity <u>of the values of f(y) that make it stationary</u>; this variation is expressed as

$$\delta I = \delta \int_a^b H\left(f(y), \frac{df(y)}{dy}, y \right) dy \qquad (5.4.3.2)$$

• under suitable continuity conditions on f and df/dy, the variation can be carried under the integral,

$$\delta I = \int_a^b \delta H\left(f(y), \frac{df(y)}{dy}, y \right) dy \qquad (5.4.3.3)$$

Euler – Lagrange equation (alternate approach)

• we know that if a function H is a function of two variables, u and v, its total differential is given by

$$dH(u,v) = \frac{\partial H}{\partial u} du + \frac{\partial H}{\partial v} dv$$

• in a completely analogous way, the variation of H when H depends on two functions u and v is given by

$$\delta H(u,v) = \frac{\partial H}{\partial u} \delta u + \frac{\partial H}{\partial v} \delta v \qquad (5.4.3.4)$$

Euler-Lagrange equation (alternate approach)

- placing (5.4.3.4) into (5.4.3.1)

$$\delta I = \int_a^b \left(\frac{\partial H}{\partial f}\delta f + \frac{\partial H}{\partial\left(\frac{df}{dy}\right)} \underbrace{\delta\left(\frac{df}{dy}\right)} \right) dy \qquad (5.4.3.5)$$

- now the derivative of the variation equals the variation of the derivative:

$$\delta\left(\frac{df}{dy}\right) = \frac{d}{dy}(\delta f)$$

- and substituting in (5.4.3.5):

$$\delta I = \int_a^b \left(\frac{\partial H}{\partial f}\delta f + \frac{\partial H}{\partial\left(\frac{df}{dy}\right)} \frac{d}{dy}(\delta(f)) \right) dy \qquad (5.4.3.6)$$

Euler-Lagrange equation (alternate approach)

- integrate the second term of the integrand in eq. (5.4.3.6) by parts by letting:

$$u = \frac{\partial H}{\partial\left(\frac{df}{dy}\right)} \Rightarrow du = \frac{d}{dy}\left[\frac{\partial H}{\partial\left(\frac{df}{dy}\right)}\right]dy$$

$$dv = \frac{d}{dy}\left(\delta(f)\right)dy \Rightarrow v = \delta(f)$$

recall,

$$\int u\,dv = uv - \int v\,du$$

- to obtain

$$\int_a^b \frac{\partial H}{\partial\left(\frac{df}{dy}\right)}\frac{d}{dy}\left(\delta(f)\right)dy = \frac{\partial H}{\partial\left(\frac{df}{dy}\right)}\delta(f)\Big]_a^b - \int_a^b \delta(f)\frac{d}{dy}\left[\frac{\partial H}{\partial\left(\frac{df}{dy}\right)}\right]dy \qquad (5.4.3.7)$$

Euler-Lagrange equation (alternate approach)

- placing (5.4.3.7) into (5.4.3.6),

$$\delta I = \frac{\partial H}{\partial\left(\frac{df}{dy}\right)}\delta\left(f\right)\Big]_a^b - \int_a^b \left(\frac{\partial H}{\partial f}\delta f - \delta f\,\frac{d}{dy}\left[\frac{\partial H}{\partial\left(\frac{df}{dy}\right)}\right]\right)dy$$

- or, setting f'=df/dy and rearranging,

$$\delta I = \frac{\partial H}{\partial f^{'}}\delta f\,\Big]_a^b - \int_a^b \left(\frac{\partial H}{\partial f} - \frac{d}{dy}\left[\frac{\partial H}{\partial f^{'}}\right]\right)\delta f dy \qquad (5.4.3.8)$$

- since now f(a) and f(b) are prescribed, their variation is zero, i.e. $\delta f(a) = \delta f(b) = 0$
and the first term of the RHS of (5.4.3.8) is zero

Euler-Lagrange equation (alternate approach)

- to minimize (or maximize) I, the variation of I must be zero; this is equivalent to saying that of all possible functions f(y) with specified values at y=a and y=b the one that makes I stationary is the one that makes its variation equal to zero; therefore, I is minimized when δI=0 or, from (5.4.3.8),

$$\delta I = 0 \Rightarrow \int_a^b \left(\frac{\partial H}{\partial f} - \frac{d}{dy}\left[\frac{\partial H}{\partial f^{'}} \right] \right) dy\,\delta f = 0 \qquad (5.4.3.9)$$

- (5.4.3.9) must be true independent of the value of δf. So:

$$\frac{\partial H}{\partial f} - \frac{d}{dy}\left[\frac{\partial H}{\partial f^{'}} \right] = 0 \qquad (5.4.3.10)$$

Euler-Lagrange eqn for I to be stationary

(note: f’=df/dy)

Euler – Lagrange equation

• if instead of one function, f(y), the integral I is in terms of two functions f(y) and g(y),

$$I = \int_a^b H\left(f(y), \frac{df(y)}{dy}, g(y), \frac{dg(y)}{dy}\, y \right) dy \qquad (5.4.3.11)$$

• an analogous procedure leads to the following two Euler-Lagrange equations

$$\left.\begin{aligned} \frac{\partial H}{\partial f} - \frac{d}{dy}\left[\frac{\partial H}{\partial f^{'}}\right] = 0 \\ \frac{\partial H}{\partial g} - \frac{d}{dy}\left[\frac{\partial H}{\partial g^{'}}\right] = 0 \end{aligned}\right\} \qquad (5.4.3.12)$$

(eqns are usually coupled)

Euler-Lagrange equation

- finally, for higher order derivatives present in the integrand

$$I = \int_a^b H\left(f(y), \frac{df(y)}{dy}, \frac{d^2 f}{dy^2}, y \right) dy$$

- the Euler-Lagrange equation takes the form

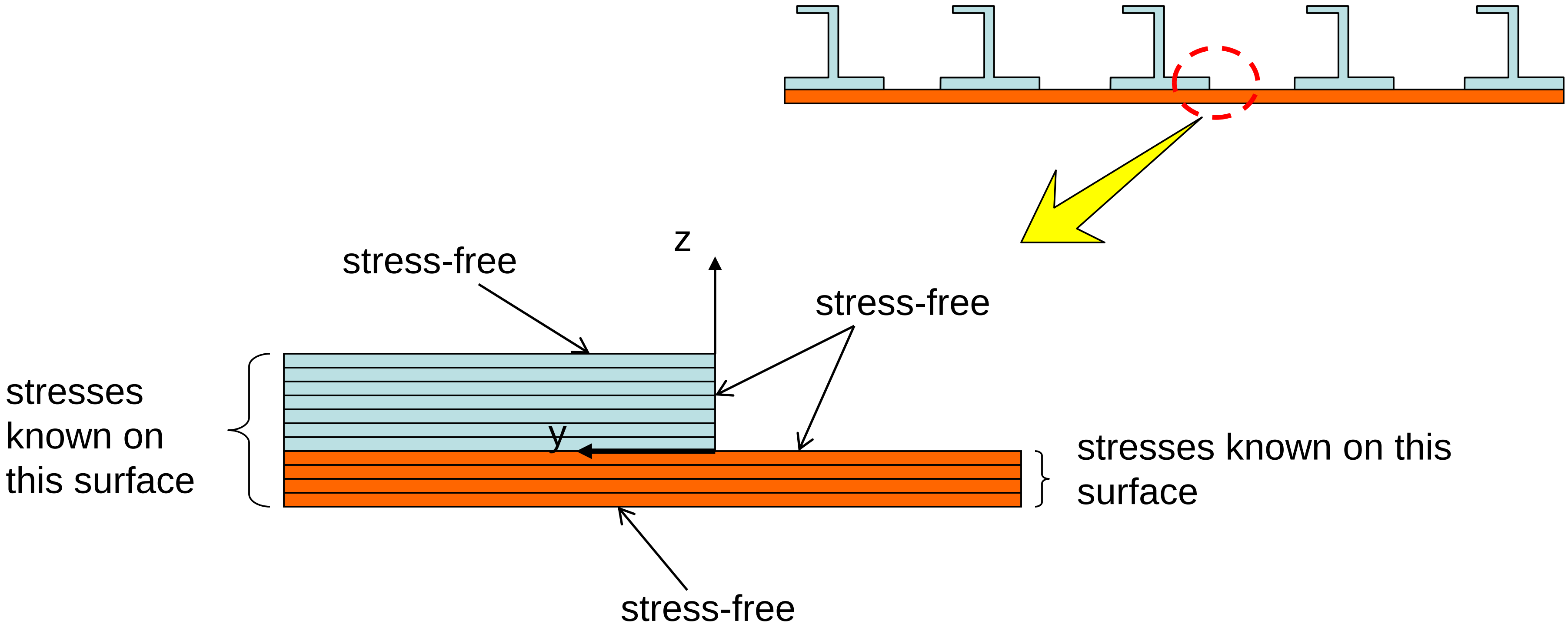
$$\frac{d^2}{dy^2}\left(\frac{\partial H}{\partial f''}\right) - \frac{d}{dy}\left(\frac{\partial H}{\partial f'}\right) + \frac{\partial H}{\partial f} = 0$$

(note: f''=d²f/dy², f''=df/dy) (5.4.3.13)

compare to (5.4.3.10)

Application to the skin-stiffener separation problem[1]

- consider the following problem



(1) Kassapoglou, C., "Stress Determination at Skin-Stiffener Interfaces of Composite Stiffened Panels Under Generalized Loading", J. of Reinforced Plastics and Composites, vol 13, 1994, pp 555-572.

Local stress calculation in skin-stiffener details

- assume the structure is long in the x direction and thus,

$$\frac{\partial}{\partial x} = 0$$

- the stress equilibrium equations then,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

- simplify to

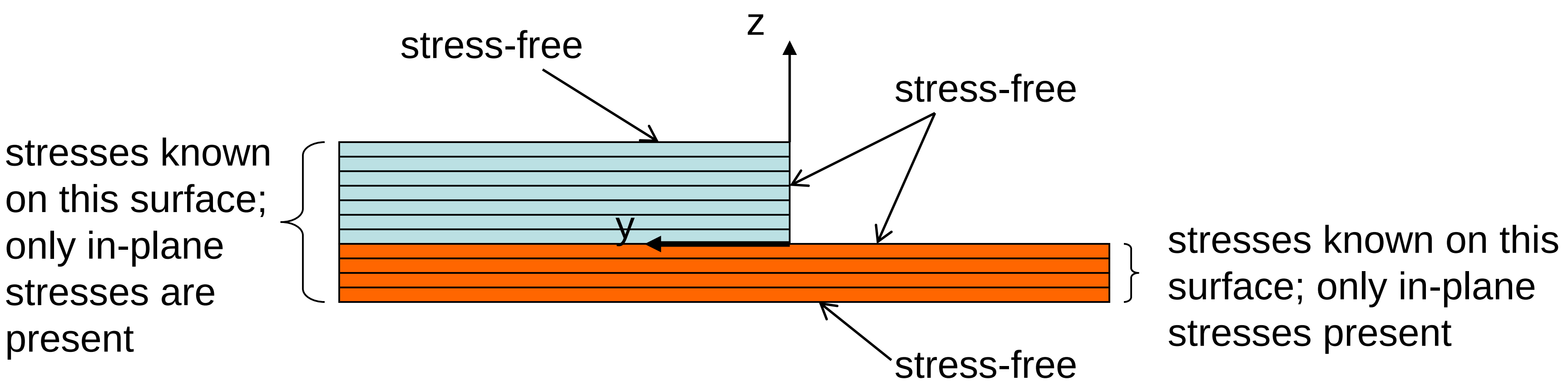
$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad (5.4.3.14)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \qquad (5.4.3.15)$$

$$\frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \qquad (5.4.3.16)$$

Local stress calculation in skin-stiffener details

- then, eq. (5.4.3.14) uncouples from the other two
- if we somehow knew two of the stresses, one from the set (τ_{xy}, τ_{xz}) and one from the set (τ_{yz}, σ_y, σ_z) we could, in principle, determine the remaining ones from the equilibrium equations



- assume also that far from the origin, the interlaminar stresses have decayed and the classical solution is recovered

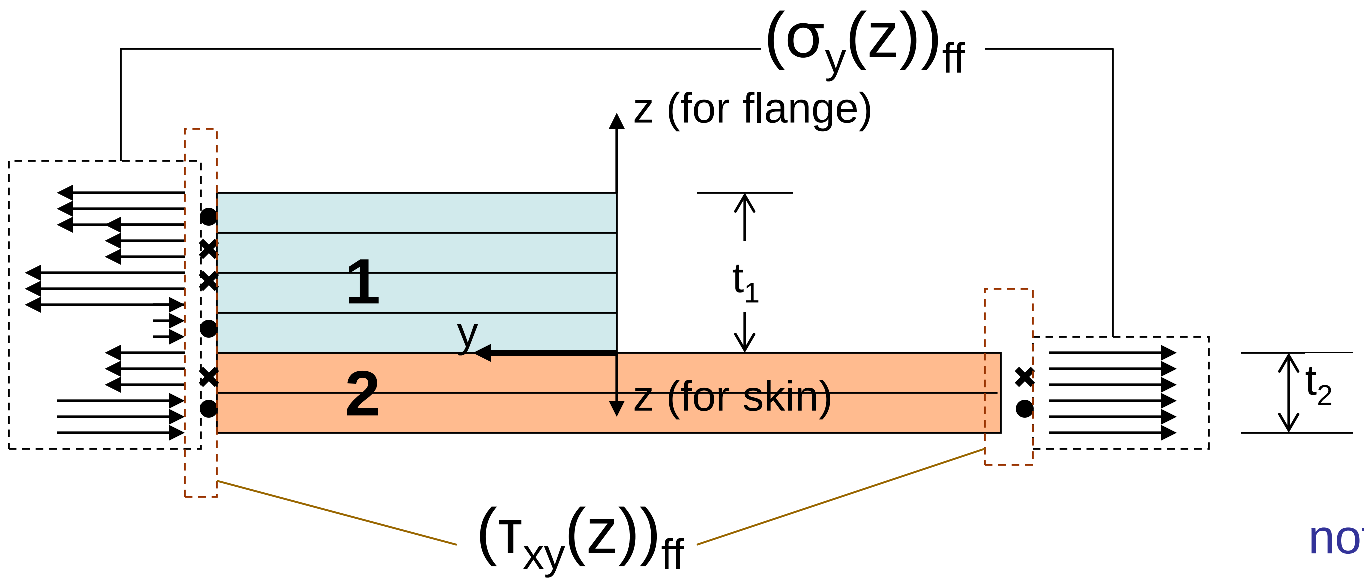
Local stress calculation in skin-stiffener details

- assume that σ_y and τ_{xy} have the form

$$\sigma_y = \left(\sigma_y(z)\right)_{ff} + f(y)F(z) \tag{5.4.3.17}$$

$$\tau_{xy} = \left(\tau_{xy}(z)\right)_{ff} + g(y)G(z) \tag{5.4.3.18}$$

$(\sigma_y(z))_{ff}$, $(\tau_{xy}(z))_{ff}$ are assumed known (CLPT solution)



note z coordinate flips

Local stress calculation in skin-stiffener details

$$\sigma_y = \left(\sigma_y(z)\right)_{ff} + f(y)F(z)$$
$$\tau_{xy} = \left(\tau_{xy}(z)\right)_{ff} + g(y)G(z)$$

- f(y) and g(y) are unknown functions
- F(z) and G(z) can be terms in a Fourier series with unknown coefficients. Truncating these series after the first term yields, **for the flange** (region 1)

$$\sigma_y = \left(\sigma_y(z)\right)_{ff} + f(y)\left[A_1 \sin\frac{\pi z}{t_1} + B_1 \cos\frac{\pi z}{t_1}\right] \qquad (5.4.3.19)$$

$$\tau_{xy} = \left(\tau_{xy}(z)\right)_{ff} + g(y)\left[C_1 \sin\frac{\pi z}{t_1} + C_2 \cos\frac{\pi z}{t_1}\right] \qquad (5.4.3.20)$$

A_1,B_1,C_1,C_2 are unknown constants

Local stress calculation in skin-stiffener details

- use (5.4.3.19) to substitute in (5.4.3.15); then

$$\frac{\partial \tau_{yz}}{\partial z} = -f'\left(A_1 \sin \frac{\pi z}{t_1} + B_1 \cos \frac{\pi z}{t_1} \right)$$

(recall ( )'=d( )/dy)

- and integrating with respect to z,

$$\tau_{yz} = -f'\left(-A_1 \frac{t_1}{\pi} \cos \frac{\pi z}{t_1} + B_1 \frac{t_1}{\pi} \sin \frac{\pi z}{t_1} \right) + P_1(y)$$

($P_1(y)$ is an unknown function)

- the top of the flange ($z=t_1$) is stress-free so $\tau_{yz}(z=t_1)=0$:

$$-f'\left(A_1 \frac{t_1}{\pi} \right) + P_1(y) = 0 \Rightarrow P_1(y) = f'\left(A_1 \frac{t_1}{\pi} \right)$$

Local stress calculation in skin-stiffener details

- substituting in the expression for τ_{yz},

$$\tau_{yz} = f'\left(A_1 \frac{t_1}{\pi}\left(1+\cos\frac{\pi z}{t_1}\right) - B_1 \frac{t_1}{\pi}\sin\frac{\pi z}{t_1}\right) \qquad (5.4.3.21)$$

- use (5.4.3.21) to substitute in (5.4.3.16); then

$$\frac{\partial \sigma_z}{\partial z} = -f''\left(A_1 \frac{t_1}{\pi}\left(1+\cos\frac{\pi z}{t_1}\right) - B_1 \frac{t_1}{\pi}\sin\frac{\pi z}{t_1}\right) \qquad (f''=d^2f/dy^2)$$

- and integrating with respect to z,

$$\sigma_z = -f''\left(A_1 \frac{t_1}{\pi}\left(z+\frac{t_1}{\pi}\sin\frac{\pi z}{t_1}\right) + B_1\left(\frac{t_1}{\pi}\right)^2 \cos\frac{\pi z}{t_1}\right) + P_2(y) \qquad (P_2(y) \text{ is an unknown function})$$

- the top of the flange ($z=t_1$) is stress-free so $\sigma_z(z=t_1)=0$:

$$-f''\left(A_1 \frac{t_1}{\pi}(t_1) + B_1\left(\frac{t_1}{\pi}\right)^2 \cos\pi\right) + P_2(y) = 0 \Rightarrow P_2(y) = f''\left(A_1 \frac{t_1^{\ 2}}{\pi} - B_1\left(\frac{t_1}{\pi}\right)^2\right)$$

Local stress calculation in skin-stiffener details

• substituting in the expression for σ_z,

$$\sigma_z = f''\left(A_1\frac{t_1}{\pi}\left(t_1 - z - \frac{t_1}{\pi}\sin\frac{\pi z}{t_1}\right) - B_1\left(\frac{t_1}{\pi}\right)^2\left(1+\cos\frac{\pi z}{t_1}\right)\right) \qquad (5.4.3.22)$$

• in a completely analogous fashion, placing (5.4.3.20) into (5.4.3.14), solving for τ_{xz} and applying the boundary condition $\tau_{xz}(z=t_1)=0$ (top of flange is stress-free) we get:

$$\tau_{xz} = g'\left[C_1\frac{t_1}{\pi}\left(1+\cos\frac{\pi z}{t_1}\right) - C_2\frac{t_1}{\pi}\sin\frac{\pi z}{t_1}\right] \qquad (5.4.3.23)$$

Local stress calculation in skin-stiffener details

- determination of σ_x
- so far σ_x was completely missing from the equations
- use the inverted stress-strain equations:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \underbrace{\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix}}_{[E]^{-1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \quad (5.4.3.24)$$

Local stress calculation in skin-stiffener details

- and the strain compatibility relations:

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} \qquad (5.4.3.25)$$

$$\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} = \frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} \qquad (5.4.3.26)$$

- but from our previous assumption of long flange in x dir,

$$\frac{\partial}{\partial x} = 0$$

- and (5.4.3.25) and (5.4.3.26) become:

$$0 = \frac{\partial^2 \varepsilon_x}{\partial y^2} \qquad (5.4.3.25a)$$

$$0 = \frac{\partial^2 \varepsilon_x}{\partial z^2} \qquad (5.4.3.26a)$$

Local stress calculation in skin-stiffener details

- use the first of eqs (5.4.3.24) to sub in (5.4.3.25a) and (5.4.3.26a):

$$\frac{\partial^2}{\partial y^2}\left[S_{11}\sigma_x + S_{12}\sigma_y + S_{13}\sigma_z + S_{16}\tau_{xy}\right] = 0 \qquad (5.4.3.25b)$$

$$\frac{\partial^2}{\partial z^2}\left[S_{11}\sigma_x + S_{12}\sigma_y + S_{13}\sigma_z + S_{16}\tau_{xy}\right] = 0 \qquad (5.4.3.26b)$$

- integrating the first twice w.r.t. y gives:

$$S_{11}\sigma_x + S_{12}\sigma_y + S_{13}\sigma_z + S_{16}\tau_{xy} = yG_1(z) + G_2(z) \qquad (5.4.3.27)$$

(recall stresses do not depend on x)

- substituting in the second,

$$y\frac{d^2G_1(z)}{dz^2} + \frac{d^2G_2(z)}{dz^2} = 0$$

Local stress calculation in skin-stiffener details

- from which,

$$G_1(z) = k_o + k_1 z$$
$$G_2(z) = k_3 + k_4 z$$

- we can now substitute in (5.4.3.27) and solve for σ_x

$$\sigma_x = K_o + K_1 y + K_2 z + K_3 yz - \frac{S_{12}}{S_{11}}\sigma_y - \frac{S_{13}}{S_{11}}\sigma_z - \frac{S_{16}}{S_{11}}\tau_{xy} \qquad (5.4.3.28)$$

- at this point, the six stresses in the flange are determined to within two unknown functions, f(y) and g(y) and a bunch on unknown coefficients

Local stress calculation in skin-stiffener details

- stress expressions in the flange:

$$\sigma_x = K_o + K_1 y + K_2 z + K_3 yz - \frac{S_{12}}{S_{11}}\sigma_y - \frac{S_{13}}{S_{11}}\sigma_z - \frac{S_{16}}{S_{11}}\tau_{xy}$$

$$\sigma_y = \left(\sigma_y(z)\right)_{ff} + f(y)\left[A_1 \sin\frac{\pi z}{t_1} + B_1 \cos\frac{\pi z}{t_1}\right]$$

$$\tau_{xy} = \left(\tau_{xy}(z)\right)_{ff} + g(y)\left[C_1 \sin\frac{\pi z}{t_1} + C_2 \cos\frac{\pi z}{t_1}\right]$$

$$\tau_{yz} = f'\left(A_1\frac{t_1}{\pi}\left(1+\cos\frac{\pi z}{t_1}\right) - B_1\frac{t_1}{\pi}\sin\frac{\pi z}{t_1}\right)$$

$$\sigma_z = f''\left(A_1\frac{t_1}{\pi}\left(t_1 - z - \frac{t_1}{\pi}\sin\frac{\pi z}{t_1}\right) - B_1\left(\frac{t_1}{\pi}\right)^2\left(1+\cos\frac{\pi z}{t_1}\right)\right)$$

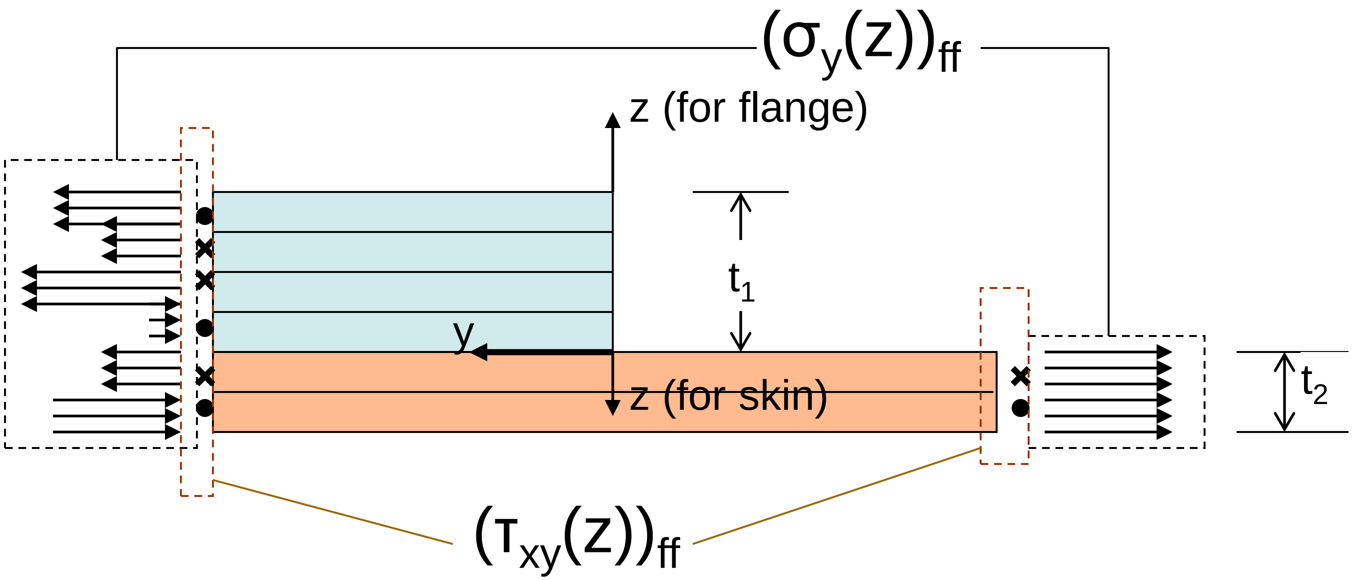
$$\tau_{xz} = g'\left[C_1\frac{t_1}{\pi}\left(1+\cos\frac{\pi z}{t_1}\right) - C_2\frac{t_1}{\pi}\sin\frac{\pi z}{t_1}\right]$$

far-field CLPT solution

interlaminar stresses

Local stress calculation in skin-stiffener details

- the solution for the skin is very similar



- require that the stresses are continuous at the flange/ skin interface (overbar denotes skin)

$$\tau_{xz}(z=0)=\bar{\tau}_{xz}(z=0)$$

$$\tau_{yz}(z=0)=\bar{\tau}_{yz}(z=0)$$

$$\sigma_{z}(z=0)=\bar{\sigma}_{z}(z=0)$$

these require that f(y) and g(y) are the <u>same</u> for skin and flange, plus eliminate some of the unknown coefficients

Energy minimization for stress calculation

- the functions f(y) and g(y) are determined by minimizing the energy

- use the complementary energy (stress-based) expression:

$$\Pi_C = \frac{1}{2}\iiint \underline{\sigma}^T \underline{S}\,\underline{\sigma}\,dydxdz + \frac{1}{2}\iiint \underline{\bar{\sigma}}^T \underline{\bar{S}}\,\underline{\bar{\sigma}}\,dydxdz - \iint \underline{T}^T \underline{u}^* \,dydz \qquad (5.4.3.29)$$

integral over the area where displacements u^* are prescribed (applied loads); T are the corresponding tractions (stresses)

- where
  - overbar denotes skin quantities
  - $\underline{\sigma}^T = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{yz} & \tau_{xz} & \tau_{xy} \end{bmatrix}$
  - $\underline{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix}$

the compliance matrix S is evaluated for the entire flange (not ply-by-ply) and for the entire skin layup

Energy minimization for stress calculation

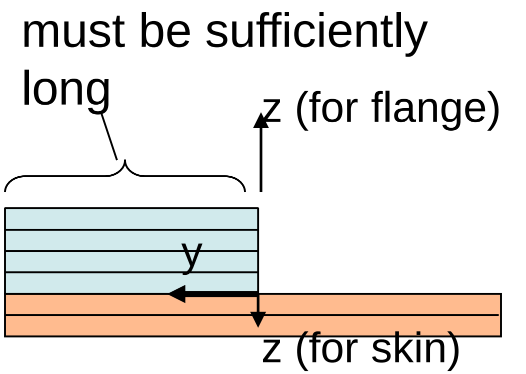
- the x and z integrations can be carried out explicitly since there is no dependence on x and the z dependence is known (to within a couple of unknown coefficients)

- carrying out the x and z integrations transforms the problem to the minimization of

$$\Pi_C = \frac{1}{2}\int H\left(\frac{d^2 f}{dy^2}, \frac{df}{dy}, f, \frac{dg}{dy}, g, y\right) dy \qquad (5.4.3.30)$$

which is <u>exactly</u> the type of integral we examined when we talked about calculus of variations

Energy minimization for stress calculation



$$\Pi_C = \frac{1}{2}\int H\left(\frac{d^2 f}{dy^2}, \frac{df}{dy}, f, \frac{dg}{dy}, g, y\right)dy$$

- A few comments:

1. Limits of integration are 0 to ∞. In reality, the upper limit can have any value as long as it is sufficiently large for the interlaminar stresses to die out (what happens for very narrow flanges?)

2. The integral has up to the second order derivative for f(y) but only up to first order derivative for g(y)

Energy minimization for stress calculation

- substituting in the expression for Π_C and using eqs. (5.4.3.12) and (5.4.3.13) leads to the following system of ODEs:

$$\frac{d^4 f}{dy^4} + R_1 \frac{d^2 f}{dy^2} + R_2 f + R_3 \frac{d^2 g}{dy^2} + R_4 g = 0$$

$$\frac{d^2 g}{dy^2} + R_5 g + R_6 \frac{d^2 f}{dy^2} + R_7 f = 0 \qquad (5.4.3.31)$$

where R_1-R_7 are constants coming from the z integration and containing the compliances S_{ij} and the coefficients in the stress expressions

Energy minimization for stress calculation

- the solution to the ODEs is:

$$f(y) = S_{1f} e^{-\phi_1 y} + S_{2f} e^{-\phi_2 y} + S_{3f} e^{-\phi_3 y}$$
$$g(y) = S_{1g} e^{-\phi_1 y} + S_{2g} e^{-\phi_2 y} + S_{3g} e^{-\phi_3 y} \qquad (5.4.3.32)$$

- with φ the solution to

$$\phi^6 + (R_1 + R_5 - R_3 R_6)\phi^4 + (R_1 R_5 + R_2 - R_3 R_7 - R_4 R_6)\phi^2 + R_2 R_5 - R_4 R_7 = 0 \qquad (5.4.3.33)$$

- and

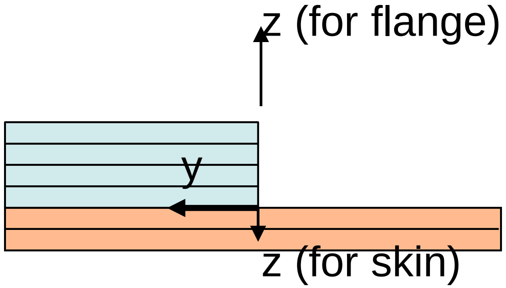
$$\frac{S_{if}}{S_{ig}} = -\frac{\phi_i^{\,2} + R_5}{R_6 \phi_i^{\,2} + R_7} \qquad (5.4.3.34)$$

- the solutions to (5.4.3.33) can be complex; only solutions with <u>positive</u> real parts are accepted!

Skin/stiffener interface stresses – remaining BC's

- at this point, the remaining boundary conditions are imposed, namely the edge of the flange is stress free:

$$\sigma_y(y=0)=0$$
$$\tau_{xy}(y=0)=0$$
$$\tau_{yz}(y=0)=0$$



- and substituting,

$$\left(S_{1f}+S_{2f}+S_{3f}\right)\left(A_1\sin\frac{\pi z}{t_1}+B_1\cos\frac{\pi z}{t_1}\right)+\left(\sigma_y(z)\right)_{ff}=0$$
$$\left(S_{1g}+S_{2g}+S_{3g}\right)\left(C_1\sin\frac{\pi z}{t_1}+C_2\cos\frac{\pi z}{t_1}\right)+\left(\tau_{xy}(z)\right)_{ff}=0 \qquad (5.4.3.35)$$
$$\left(\phi_1S_{1f}+\phi_2S_{2f}+\phi_3S_{3f}\right)\left(A_1\frac{t_1}{\pi}\left(1+\cos\frac{\pi z}{t_1}\right)-B_1\frac{t_1}{\pi}\sin\frac{\pi z}{t_1}\right)=0$$

◯ : unknown

$A_1=C_1=1$

Skin/stiffener interface stresses – remaining BC's

- the far-field (CLPT) stresses are, usually, piecewise linear in z
- by expanding the far-field stresses in Fourier series and taking the first terms, a system of 5 equations in the 5 unknowns S_{1f}, S_{2f}, S_{3f}, B_1, C_2 is obtained
- after all this, there is still, one unknown coefficient coming from the stress expressions in the skin; it is determined again by minimizing the energy
- the solution requires some iterations: a value of the unknown coefficient is assumed, all other unknowns are determined and a corrected value of the remaining unknown is determined; after a few iterations the process converges