System Identification & Parameter Estimation

Wb2301: SIPE lecture 3
Impulse and frequency response functions

Alfred C. Schouten, Dept. of Biomechanical Engineering (BMechE), Fac. 3mE
2/16/2010
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• Estimation of Impulse response functions (IRF)

• Estimation of Frequency response functions (FRF)

• Improving the estimate of spectral densities
  • ‘Welch’ method
  • Frequency averaging

• Open-loop vs. closed-loop: causality!

• Estimation of linearity: coherence functions
Basic identification with cross-covariance

\[ y(t) = n(t) + \int h(t') u(t - t') dt' \]

multiply with \( u(t-\tau) \):

\[ u(t-\tau) y(t) = u(t-\tau) n(t) + \int h(t') u(t-\tau) u(t-t') dt' \]

\[ C_{uy}(\tau) = C_{un}(\tau) + \int h(t') C_{uu}(\tau-t') dt' \]

white noise:

\[ C_{uu}(\tau) = 0 \text{ for } \tau \neq 0; \quad C_{uu}(0) = 1 \]

\[ C_{uy}(\tau) = C_{un}(\tau) + h(\tau) \]

Other 'tricks' needed when \( u(t) \) is not white
Impulse response function

- Impulse response function $h(t)$
  - reaction (output) of a system in time after an impulse

- Impulse, or dirac:
  
  \[
  \delta(x) = \begin{cases} 
  \infty, & x = 0 \\
  0, & x \neq 0 
  \end{cases}
  \]

  \[
  \int_{-\infty}^{\infty} \delta(x)dx = 1
  \]

- Output with other input
  - Convolution with $h(t)$

  \[
  y(t) = \int h(\tau)u(t-\tau)d\tau
  \]
Impulse response function

- Causal system: \( h(t) = 0 \) for \( t < 0 \)
- Finite memory: \( h(t) = 0 \) for \( t > T \)

\[
y(t) = \int_0^T h(\tau) u(t - \tau) d\tau
\]

- Discrete

\[
y(t) = \sum_{\tau=0}^{T-1} h(\tau) u(t - \tau) \Delta \tau
\]
Estimation of impulse response function (IRF)

- Westwick & Kearney, p. 106-115
  - Direct estimation
  - Least-squares regression
  - Correlation-based methods
Direct Estimation of impulse response function (IRF)

• Apply (multiple) impulses
  • True impulse is physically impossible!
  • Alternative: pulse with fixed width and height

• Disadvantages
  • Impractical
  • Amplitude constraints
  • Noise
IRF via least squares regression

\[ y(t) = \sum_{\tau=0}^{T-1} h(\tau) u(t - \tau) \Delta \tau \]

- Rewrite convolution as matrix

\[ y = Uh \]

with

\[
U = \begin{bmatrix}
u(1) & 0 & 0 & \ldots & 0 \\
u(2) & u(1) & 0 & \ldots & 0 \\
u(3) & u(2) & u(1) & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
u(N) & u(N-1) & u(N-2) & \ldots & u(N-T+1)
\end{bmatrix}
\]
IRF via least squares regression

- $z$ is measurement of $y$, with noise $n(t)$:
  
  $$z(t) = y(t) + \nu(t)$$

  $$z = Uh + \nu$$

- Solution via linear least-squares regression (see W&K p.26):
  
  $$\hat{h} = (U^T U)^{-1} U^T z$$
IRF via correlation-based methods

- See Westwick and Kearney, not discussed
Frequency Domain Expressions

- **Discrete Fourier Transform:**
  \[
  U(f) = \mathcal{F}(u(t)) = \sum_{t=1}^{N} u(t) e^{-j2\pi \frac{ft}{N}}
  \]

  where \( f \) takes values 0, 1, ..., \( N-1 \) multiples of \( \Delta f = \frac{1}{N\Delta t} \)

- **Inverse Fourier Transform:**
  \[
  u(t) = \mathcal{F}^{-1}(U(f)) = \frac{1}{N} \sum_{f=1}^{N} U(f) e^{j2\pi \frac{ft}{N}}
  \]
Frequency domain models

• Time-domain: convolution integral

\[ y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau \]

• ‘Convolution in time-domain is multiplication in frequency domain’ (and vice versa)

frequency domain:  \[ Y(f) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau \right) e^{-2\pi f \tau} dt \]

\[ = \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} u(t-\tau) e^{-2\pi f \tau} dt d\tau \]

\[ = U(f) \int_{-\infty}^{\infty} h(\tau) e^{-2\pi f \tau} d\tau \]

\[ = U(f) H(f) \]
Estimation of frequency response function (FRF)

- Sinusoidal frequency response testing
  - Apply single sinusoid

- Stochastic frequency response testing
  - Noise based

- Periodic frequency response testing
  - Pintelon & Schoukens, discussed in Lecture 4
Basic identification with spectral densities

\[ y(t) = n(t) + \int h(t') u(t - t') \, dt' \]

multiply with \( u(t - \tau) \):

\[ u(t - \tau) y(t) = u(t - \tau) n(t) + \int h(t') u(t - \tau) u(t - t') \, dt' \]

\[ C_{uy}(\tau) = C_{un}(\tau) + \int h(t') C_{uu}(\tau - t') \, dt' \]

Fourier transform:

\[ S_{uy}(f) = S_{un}(f) + \tilde{H}(f) S_{uu}(f) \]

if \( S_{un}(f) = 0 \):

\[ \tilde{H}(f) = \frac{S_{uy}(f)}{S_{uu}(f)} \]
Spectral densities (lecture 2)

• **Spectral density**
  - Defined as Fourier transform of cross-correlation (indirect approach)
  
  \[
  \hat{S}_{uy}(f) = \sum_{\tau=0}^{N-1} \hat{\Phi}_{uy}(\tau) e^{-j2\pi \frac{f\tau}{N}}
  \]

  - Only evaluated for till \( f_s/2 \); \( S_{uy}(-f) = S_{uy}(f)^* \)
  - Direct approach via transformed signals

\[
\hat{S}_{uy}(f) = \frac{1}{N} U^*(n\Delta f) Y(n\Delta f)
\]

• **Properties auto-spectral density**
  - Real valued (no imaginary part)
  - Parceval: area under \( S_{uu} \) is related to signal’s variance

• **Properties cross-spectral density**
  - Complex values
  - Gives interdependency between two signals (gain&phase)
Properties of the spectral density estimator

- Matlab demo: Lec3_length_Suu.m
  Relation between number of samples and variance of the estimator

- Increasing the number of samples:
  - Longer observation: increased frequency resolution
  - Higher sample frequency: increased frequency bandwidth
  - \( \Rightarrow \) Variance of the (raw) estimator remains equal!

\( \Rightarrow \) The raw estimator is not consistent

- The raw estimate for the auto-spectral density is sometimes called the periodogram
Improving the estimate

- Common techniques to improve the spectral estimate
  - Frequency averaging
  - Welch method

- Other (=old) methods:
  - Multiply cross-covariance with window before DFT
  - Convolve the spectral density with window
  - Will be discussed during Lecture 4

- Not to confuse with signal windowing (also ‘tapering’); is done before DFT, will be discussed during Lecture 4
Improving the estimate

• Reduce variance of spectral estimator by averaging
  • either over multiple repetitions,
  • or over adjacent frequencies

• Basic ‘idea’:
  • Multiple repetitions: each realizations has the same frequency content.
  • Frequency averaging: the spectral density is often smooth, i.e. adjacent frequencies contain (approx.) the same information.

• => Averaging will reduce the effect of noise, as the noise has zero mean.
Welch method (averaged periodogram)

- Divide data in multiple segments
- Calculate spectral density for each segment
- Average over the segments

\[
\hat{S}_{uu}(f) = \frac{1}{D} \sum_{d=1}^{D} S_{uu}(f) = \frac{1}{DN_D} \sum_{d=1}^{D} U(-f)U(f)
\]
Frequency averaging

- Calculate the raw spectral density
- Average over adjacent frequencies

\[ \hat{S}_{uu}(f_c) = \frac{1}{D} \sum_{d=1}^{D} \hat{S}_{uu}(f_d); \quad f_c = \frac{1}{D} \sum_{d=1}^{D} f_d \]

- Possible drawback: can introduce bias when the power at adjacent frequencies is not similar (e.g. sharp oscillatory peaks)

=> Matlab demo: Lec3_SmoothSuu.m
Frequency response of some basic systems

- Example frequency response functions of different systems (Matlab demo: Lec2_example_systems.m):
  - Time delay
  - 1st order
  - 2nd order

- Frequency response functions (FRF) of different systems:
  - Conclusion: the system can ‘easy’ be recognized from a frequency response functions (system order, natural frequency, relative damping, etc)
time delay

1st order system

2nd order system

gain [-]

phase [°]

frequency [Hz]
Example FRF

- Admittance of human arm
Coherence and coherency

- Coherency
  \[ \gamma_{uy}(f) = \frac{S_{uy}(f)}{\sqrt{S_{yy}(f)S_{uu}(f)}} \]

- Coherence
  \[ \gamma_{uy}^2(f) = \frac{|S_{uy}(f)|^2}{S_{yy}(f)S_{uu}(f)} \]

- Coherence
  - Real valued, between 0 and 1
  - Squared coherency

- Coherency has a phase, which represents the relative delay between the signals
Time-domain vs. Frequency-domain

**Time Domain**
- Input, output: $x(t), y(t)$
- Cross-correlation function: $\Phi_{xy}(\tau)$
- Cross-covariance function: $C_{xy}(\tau)$
- Correlation coefficient: $r_{xy}(\tau)$

**Fourier Transformation**

**Frequency Domain**
- Input, output: $X(f), Y(f)$
- Cross-spectral density: $S_{xy}(f)$
- Coherency: $\gamma_{xy}(f)$

Fourier Transformation
Coherence $\gamma_{uy}^2$

Coherence:

$$\gamma_{uy}^2(f) = \frac{|S_{uy}(f)|^2}{S_{yy}(f)S_{uu}(f)} \quad 0 \leq \gamma_{yu}^2(f) \leq 1$$

Coherence: Linear relationship between input and output, irrespective of the type of system in between.
Theoretical open loop coherence

\[ S_{uy}(f) = H(f)S_{uu}(f) + S_{un}(f) \]

if \( S_{un}(f) = 0 \);

\[ S_{yy}(f) = \left| H(f) \right|^2 S_{uu}(f) + S_{nn}(f) \]

\[ \gamma_{uy}^2(f) = \frac{\left| S_{uy}(f) \right|^2}{S_{yy}(f)S_{uu}(f)} = \frac{\left| H(f)S_{uu}(f) \right|^2}{S_{uu}(f)\left( \left| H(f) \right|^2 S_{uu}(f) + S_{nn}(f) \right)} \]

\[ \gamma_{uy}^2(f) = \frac{1}{1 + \frac{S_{nn}(f)}{\left| H(f) \right|^2 S_{uu}(f)}} \]
Coherence $\gamma_{uy}^2$

$$\hat{\gamma}_{uy}^2(f) = \frac{|\hat{S}_{uy}(f)|^2}{\hat{S}_{yy}(f)\hat{S}_{uu}(f)} = \frac{\left| \frac{1}{N}U(-f)Y(f) \right|^2}{\frac{1}{N}Y(-f)Y(f) \cdot \frac{1}{N}U(-f)U(f)}$$

$$\hat{\gamma}_{uy}^2(f) = \frac{|U(-f)Y(f)|^2}{|Y(f)|^2 \cdot |U(f)|^2} = 1$$

- With the ‘raw’ spectra the coherence equals 1!
- Smoothing is required. However as a result of the squared cross-spectrum the coherence will be overestimated (bias)
Coherence

- Coherence indicates if two signals are linearly related
  - Reduced by additional signals (noise) and nonlinearities

- Raw estimate of coherence is always 1
  - Artifact!

- Smoothing of spectral densities is required to get a realistic estimate

- Effect of smoothing on estimator for the coherence
  - Coherence estimator is always biased; overestimated as a result of the square in the coherence (structural error => bias)
  - With averaging the estimator approaches the ‘true’ value
Examples from neuroscience

• Corticomuscular coherence (CMC)
  • Calculated between EEG and EMG. It is thought to represent a functional connection between brain (motor cortex) and muscles. Best found during isometric contractions in subjects.

• Intermuscular coherence
  • Calculated between EMGs of different muscles. It indicates that muscles are driven by one common ‘drive’. Found in specific motor disorders (o.a. myoclonus dystonia). Normally each muscle is activated by a specific area in the motor cortex, as such no significant coherence exists.
Corticomuscular coherence (CMC)

- Example CMC
Significance of coherence

• With corticomuscular or intermuscular coherence, one is interested if coherence exists, i.e. is it significantly different from zero coherence (no coherence)

• If coherence is higher than significance level, signals are linearly related (often very weak, as coherence is between 0.1-0.3)

• Note some authors present coherency (and other do not even explicitly mention what is presented)
  • If coherence = 0.1 - 0.3, than coherency = 0.32 - 0.55)
Summary: effect of Welch method or frequency averaging

- Estimators for the spectral densities
  - Variance decreases with averaging
  - Resolution decreases with averaging!

- Effect on estimator of FRF
  - Variance decreases with averaging
  - Able to estimate at higher frequencies (where normally the noise would deteriorate the estimate)

- Effect on estimator for the coherence
  - Coherence estimator is always biased
  - With averaging the estimator approaches the ‘true’ value
Identification of open loop systems

- **Open loop identification in frequency domain**
  - In most cases the input and the noise are not correlated
    \( (S_{nu}(f) = 0 \text{ for all } f) \)
  
- Estimator:
  \[
  \hat{H}(f) = \frac{\hat{S}_{uy}(f)}{\hat{S}_{uu}(f)} \quad (\hat{\text{^ denotes estimate}})
  \]
Identification of open loop system II

\[ Y(\omega) = H(\omega).U(\omega) + N(\omega) \]

Additional signal \( Z(\omega) \):


\[ S_{zy}(\omega) = H(\omega).S_{zu}(\omega) + S_{zn}(\omega) \]

if \( Z(\omega) \) is uncorrelated with \( N(\omega) \):

\[ S_{zn}(\omega) = 0 \]

\[ S_{zy}(\omega) = H(\omega).S_{zu}(\omega) \]

if \( U(\omega) \) is uncorrelated with \( N(\omega) \), then \( Z(\omega) = U(\omega) \):

\[ S_{uy}(\omega) = H(\omega).S_{uu}(\omega) \]

\[ H(\omega) = S_{uy}(\omega)/S_{uu}(\omega) \]
Causality and cross-covariance

- Demo in Matlab: Lec3_example_causelity_Cyu.m
Basic theory: causality

- Physical systems are causal: output depends on *previous* values of the input.

- Anti-causal: depends on *only future* values of the input.
  => input-output are exchanged

- Non-causal: depends on previous and future values of the input.
  => closed-loop systems
  => two parallel subsystems
Identification of a system in closed loop

- Closed loop: $S_{un}(f) \neq 0$
- Consequently: $H(f) \neq S_{uy}(f)/S_{uu}(f)$
Identification of a system in closed loop

- Closed loop: $S_{un}(f) \neq 0$
- Consequently: $H(f) \neq \frac{S_{uy}(f)}{S_{uu}(f)}$
Examples of closed loop identification

- Human-machine interaction (driving, steering, etc.)
  - Interaction between two systems!
- Human motion control
  - Muscle force depends on activation, activation depends on reflexes, reflexes depend on movement.
- Chemical/nuclear plants
  - Plant is unstable, so a controller is needed.
  - Identification around a desired operation point, a controller is required to keep the system in the desired operation point.
• What happens if we would use an open loop estimator for a system in closed loop:

\[ G_1'(f) = \frac{S_{uy}(f)}{S_{uu}(f)} \]

• What is relation between \( G_1' \) and true \( G_1 \)
Readings

• Book Westwick & Kearney
  • Chapter 1, all (lecture 1)
  • Chapter 2, sec. 2.1 – 2.3.4 (lecture 1+2)
  • Chapter 3, sec. 3.1 – 3.2 (lecture 2)
  • Chapter 5, sec. 5.1 – 5.3 (lecture 3)

• Book Pintelon & Schoukens
  • Chapter 1, sec 1.1 – 1.4 (optional, lecture 1)
  • Chapter 2, all (lecture 4)
  • Chapter 4, all (lecture 4)

• Articles
  • de Vlugt et al. (lecture 5)