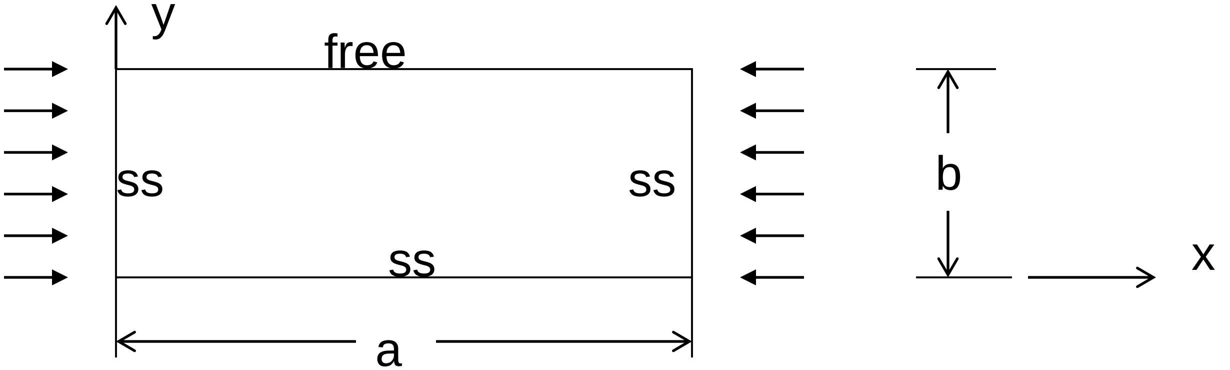
Uni-axial compression; Other BC's

- Of particular interest (for use with stiffeners):



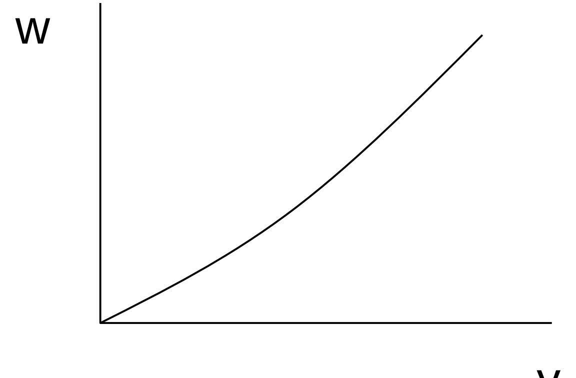
- Use:

choice of λ is important!

$$w = \sum\sum A_{mn} \sin\frac{m\pi x}{a} \sin\frac{\lambda n\pi y}{b}$$

- to substitute in governing equation:

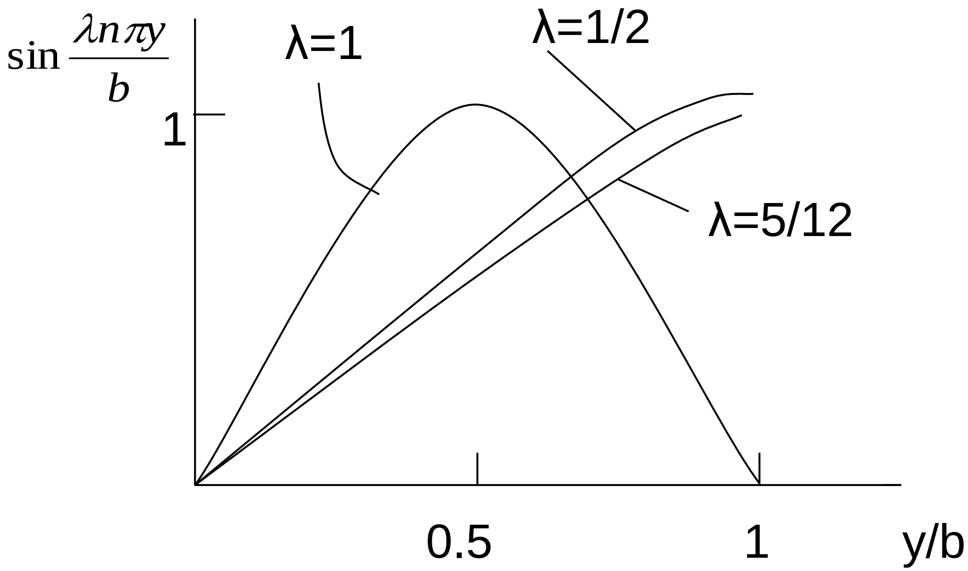
$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = N_x \frac{\partial^2 w}{\partial x^2}$$



Buckling shape versus y

Uni-axial compression; 3 sides ss, one unloaded side free

- λ is chosen so that the “free” condition is represented (e.g. if λ=1=> simply-supported)



Uni-axial compression; 3 sides ss, one unloaded side free

- Current choice of w does not satisfy all BC's:

$$w(x=0)=w(x=a)=0$$ OK

$$w(y=0)=0$$ OK

$$M_x=-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2}=0 \quad at \quad x=0,a$$ OK

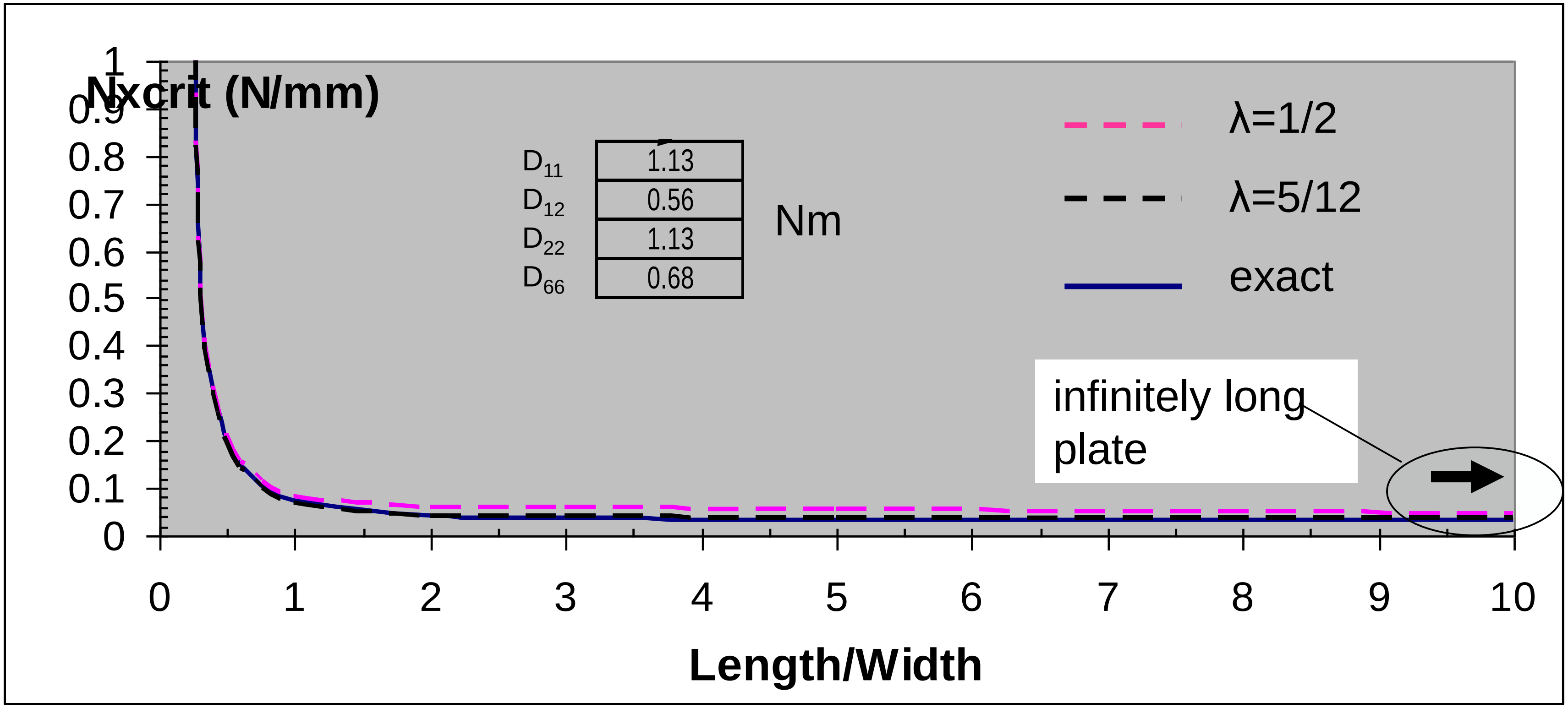
$$M_y=-D_{12}\frac{\partial^2 w}{\partial x^2}-D_{22}\frac{\partial^2 w}{\partial y^2}=0 \quad at \quad y=0,b$$???!!

- Expression for the buckling load

$$N_o=\frac{\pi^2\left[D_{11}m^4+2(D_{12}+2D_{66})m^2\lambda^2(AR)^2+D_{22}(AR)^4\lambda^4\right]}{a^2m^2}$$

Uni-axial compression; 3 sides ss, one unloaded side free

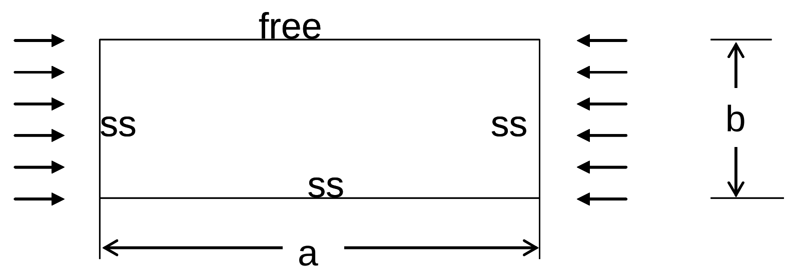
- Comparison with exact solution



Infinitely long plate, 3 sides ss, one unloaded side free

- exact solution:

$$N_{xcrit} = \frac{12D_{66}}{b^2}$$



- approximate solution:

$$N_{xcrit} = \frac{4\pi^2}{b^2}\lambda^2 D_{66} + \frac{2\pi^2}{b^2} D_{12}$$

$\approx$ 6.9-9.9

D_{11}	1.13
D_{12}	0.56
D_{22}	1.13
D_{66}	0.68

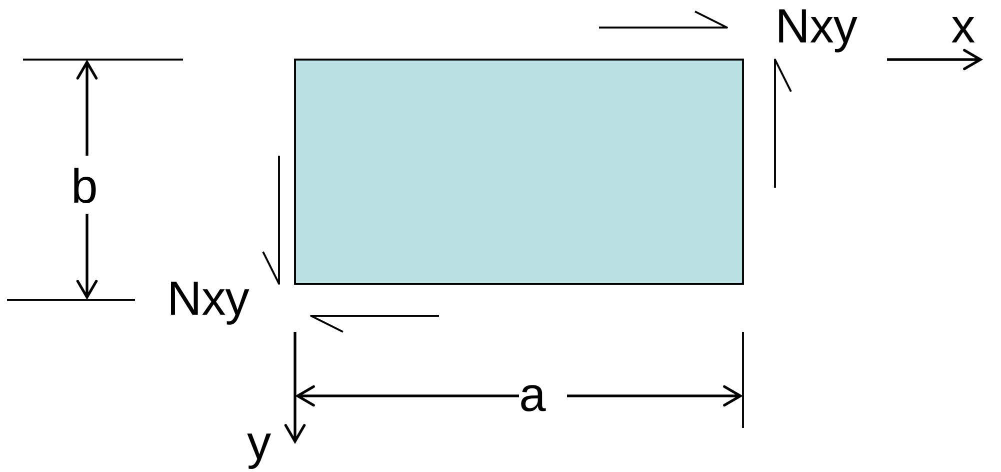
Nm

b=50.8 cm

exact: N_{xcrit}=0.032 N/mm		
λ	N_{xcrit}	Δ(%)
1/2	0.047	46.9
5/12	0.036	12.5

Buckling of rectangular plate under shear

- simply supported plate*



$D_{16}=D_{26}=B_{ij}=0$

*Whitney, J.M., *Structural Analysis of Laminated Anisotropic Plates*, Technomic Publishing, 1987, section 5.7

Buckling under shear (cont'd)

- Galerkin solution
- Governing equation:

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=2N_{xy}\frac{\partial^2 w}{\partial x\partial y}$$

- assume solution w in the form:

$$w=\sum\sum A_{mn}\underbrace{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}_{\text{characteristic functions}}$$

characteristic functions

- multiply governing equation by the characteristic functions, integrate over the domain of the plate and set the result equal to zero (satisfied for m,n=1,2,…)

$$\iint\left[D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}-2N_{xy}\frac{\partial^2 w}{\partial x\partial y}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy=0$$

Buckling under shear (cont'd)

- Substituting for w and carrying out the integrations:

$$\pi^4\left[D_{11}m^4+2(D_{12}+2D_{66})m^2n^2(AR)^2+D_{22}n^4(AR)^4\right]A_{mn}-32mn(AR)^3b^2N_{xy}\sum\sum T_{ij}A_{ij}=0$$

$$T_{ij}=\frac{ij}{(m^2-i^2)(n^2-j^2)}\quad for\quad m\pm i\quad odd\quad and\quad n\pm j\quad odd$$

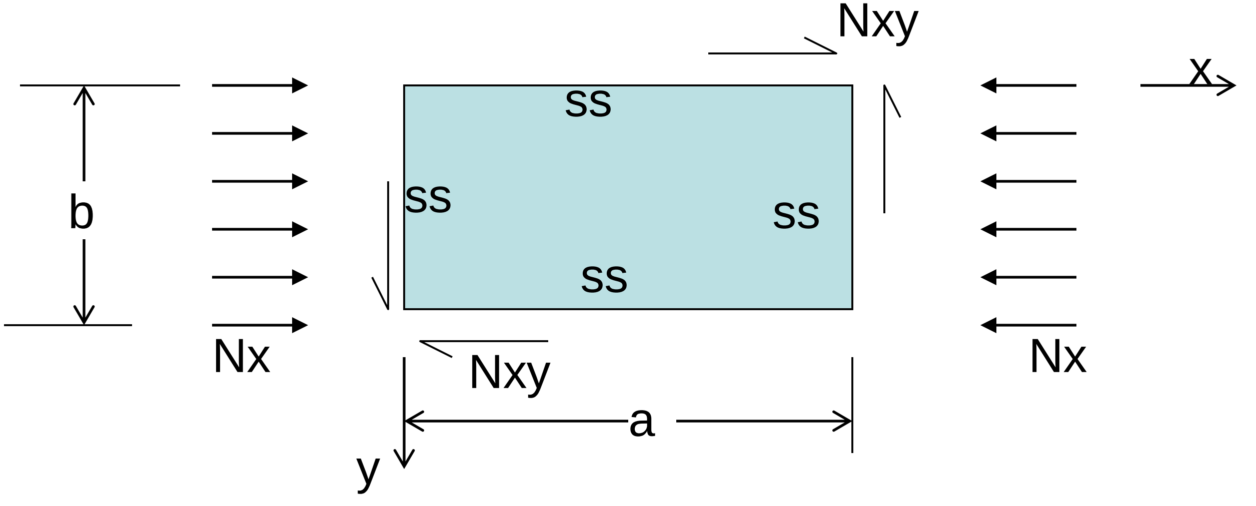
(AR)=a/b, aspect ratio

$$T_{ij}=0\quad otherwise$$

- Equation breaks down to two sets of homogeneous equations, one for m+n odd and one for m+n even; each is of the form: [**E**]{$\mathbf{A_{mn}}$}= N_{xy} [**H**] {$\mathbf{A_{mn}}$}
- => two generalized eigenvalue problems
- Lowest eigenvalue corresponds to buckling load; note that for specially orthotropic plates the eigenvalues are in pairs of positive and negative numbers of same magnitude

Buckling under combined loads

- Combined compression and shear, simply-supported all around



Buckling under compression and shear

- approximate solution (2 terms):

$$w = w_1 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + w_2 \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}$$

Single term won't work for shear!

- satisfies displacement (and force) BC's:

$$\mathrm{w(x = 0) = w(x = a) = 0}$$

$$\mathrm{w(y = 0) = w(y = b) = 0}$$

$$M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at x=0,a}$$

$$M_y = -D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at y=0,b}$$

no need to satisfy force BC's for energy minimization approach (but need more terms)

- w_1, w_2 are unknown

Energy minimization

- Minimize:

$$\Pi_c = \frac{1}{2}\iint \left\{ D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + 4D_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial y} + 4D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x \partial y} \right\} dxdy +$$

$$\frac{1}{2}\iint N_x \left(\frac{\partial w}{\partial x}\right)^2 dxdy + \iint N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} dxdy$$

  - integrations carried over the entire plate
  - Nx, Nxy constant
  - $\frac{N_{xy}}{N_x} = k$
  - $D_{16}=D_{26}=0$

Some intermediate results

$$\left(\frac{\partial^2 w}{\partial x^2}\right)^2 = w_1^{\;2}\frac{\pi^4}{4b^4}\left(1-\cos\frac{2\pi x}{a}\right)\left(1-\cos\frac{2\pi y}{b}\right) + w_2^{\;2}\frac{16\pi^4}{4b^4}\left(1-\cos\frac{4\pi x}{a}\right)\left(1-\cos\frac{4\pi y}{b}\right) +$$

$$2w_1 w_2\frac{4\pi^4}{b^4}\frac{1}{4}\left(\cos\frac{\pi x}{a}-\cos\frac{3\pi x}{a}\right)\left(\cos\frac{\pi y}{b}-\cos\frac{3\pi y}{b}\right)$$

… similarly for other integrands

- carrying out the integrations:

$$\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dxdy = w_1^{\;2}\frac{\pi^4}{4a^4}ab + w_2^{\;2}\frac{4\pi^4}{a^4}ab$$

$$\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial y^2}\right)^2 dxdy = w_1^{\;2}\frac{\pi^4}{4b^4}ab + w_2^{\;2}\frac{4\pi^4}{b^4}ab$$

$$\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) dxdy = w_1^{\;2}\frac{\pi^4}{4a^2b^2}ab + w_2^{\;2}\frac{4\pi^4}{a^2b^2}ab$$

$$\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dxdy = w_1^{\;2}\frac{\pi^2}{4a^2}ab + w_2^{\;2}\frac{\pi^2}{a^2}ab$$

$$\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 dxdy = w_1^{\;2}\frac{\pi^4}{4a^2b^2}ab + w_2^{\;2}\frac{4\pi^4}{a^2b^2}ab$$

$$\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) dxdy = \frac{w_1 w_2\pi^2}{2ab}\left(\frac{2a}{3\pi}+\frac{2a}{\pi}\right)\left(\frac{2b}{3\pi}-\frac{2b}{\pi}\right) + \frac{w_1 w_2\pi^2}{2ab}\left(\frac{2a}{3\pi}-\frac{2a}{\pi}\right)\left(\frac{2b}{3\pi}+\frac{2b}{\pi}\right)$$

Final energy expression

$$\Pi_c = \frac{1}{2}\left\{ \begin{array}{l} D_{11}\left[w_1^{\;2}\dfrac{\pi^4}{4a^3}b + w_2^{\;2}\dfrac{4\pi^4}{a^3}b \right] + 2(D_{12}+2D_{66})\left[w_1^{\;2}\dfrac{\pi^4}{4ab} + w_2^{\;2}\dfrac{4\pi^4}{ab} \right] + \\ D_{22}\left[w_1^{\;2}\dfrac{\pi^4}{4b^3}a + w_2^{\;2}\dfrac{4\pi^4}{b^3}a \right] \end{array} \right\} -$$

$$\frac{N_o}{2}\left[w_1^{\;2}\frac{\pi^2}{4a}b + w_2^{\;2}\frac{\pi^2}{a}b \right] - kN_o w_1 w_2\left(-\frac{32}{9} \right)$$

$N_o = N_{xcrit} = -N_x$

- to determine w_1, w_2,

$$\frac{\partial \Pi_c}{\partial w_1} = 0$$

$$\frac{\partial \Pi_c}{\partial w_2} = 0$$

System of equations

- which lead to

$$\frac{1}{2}\left\{D_{11}\frac{w_1\pi^4 b}{2a^3}+2(D_{12}+2D_{66})\frac{\pi^4 w_1}{2ab}+D_{22}\frac{w_1\pi^4 a}{2b^3}\right\}-N_o\frac{w_1\pi^2 b}{4a}+\frac{32}{9}kN_o w_2=0$$

$$\frac{1}{2}\left\{D_{11}\frac{8w_2\pi^4 b}{a^3}+2(D_{12}+2D_{66})\frac{8\pi^4 w_2}{ab}+D_{22}\frac{8w_2\pi^4 a}{b^3}\right\}-N_o\frac{w_2\pi^2 b}{a}+\frac{32}{9}kN_o w_1=0$$

- homogeneous system of two eqns in the two unknowns w_1, w_2
- trivial solution $w_1=w_2=0$ corresponds to in-plane deformations of the plate

System of equations

- Setting

$$K_1 = \frac{1}{4}\left[D_{11}\frac{\pi^4 b}{a^3} + 2(D_{12} + 2D_{66})\frac{\pi^4}{ab} + D_{22}\frac{\pi^4 a}{b^3} \right]$$

- and using matrix notation

eigen value

$$\underbrace{\begin{bmatrix} K_1 & 0 \\ 0 & 16K_1 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}}_{\underline{x}} = \underbrace{N_o}_{\alpha} \underbrace{\begin{bmatrix} \frac{\pi^2 b}{4a} & -\frac{32}{9}k \\ -\frac{32}{9}k & \frac{\pi^2 b}{a} \end{bmatrix}}_{\underline{B}} \underbrace{\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}}_{\underline{x}}$$

generalized eigen-value problem of the form:

$$\underset{\sim}{A}\,\underset{\sim}{x} = \alpha\,\underset{\sim}{B}\,\underset{\sim}{x}$$

Solution to the eigenvalue problem

- pre-multiply both sides of the equation by $\underline{B}^{-1}$

$$\underbrace{\underset{\sim}{B}^{-1}\underset{\sim}{A}}_{\underset{\sim}{C}}\underset{\sim}{x} = \alpha \underbrace{\underset{\sim}{B}^{-1}\underset{\sim}{B}}_{\underset{\sim}{I}}\underset{\sim}{x}$$

- obtain standard eigen value problem:

$$\underset{\sim}{C}\,\underset{\sim}{x} = \alpha\,\underset{\sim}{x}$$

- the eigenvalues are obtained as solutions to:

$$\det\left\lfloor \underset{\sim}{C} - \alpha\,\underset{\sim}{I} \right\rfloor = 0$$

For our specific example

$$\underset{\sim}{B}^{-1} = \frac{1}{\frac{\pi^4 b^2}{4a^2} - \left(\frac{32}{9}k\right)^2} \begin{bmatrix} \frac{\pi^2 b}{a} & \frac{32}{9}k \\ \frac{32}{9}k & \frac{\pi^2 b}{4a} \end{bmatrix}$$

and after some rearranging, the standardized eigenvalue problem has the form,

$$\begin{bmatrix} \frac{\pi^2 b}{a} & 16\frac{32}{9}k \\ \frac{32}{9}k & \frac{\pi^2 b}{4a}16 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = N_o\left(\frac{\pi^4 b^2}{4a^2} - \left(\frac{32}{9}k\right)^2\right)\frac{1}{K_1} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}$$

eigenvalue α

Determination of buckling load(s)

- the eigenvalue is the solution to:

$$\left(\frac{\pi^2 b}{a} - \alpha\right)\left(\frac{4\pi^2 b}{a} - \alpha\right) - \frac{512(32)}{81} k^2 = 0$$

- solving for α and substituting in terms of N_o:

$$N_o = \frac{\pi^2}{a^2} \frac{\left(D_{11} + 2(D_{12} + 2D_{66})\frac{a^2}{b^2} + D_{22}\frac{a^4}{b^4}\right)}{2 - \frac{8192}{81}\frac{a^2}{b^2\pi^4}k^2} \left[5 \pm \sqrt{9 + \frac{65536}{81}\frac{a^2}{\pi^4 b^2}k^2}\right]$$

two solutions: use the lowest

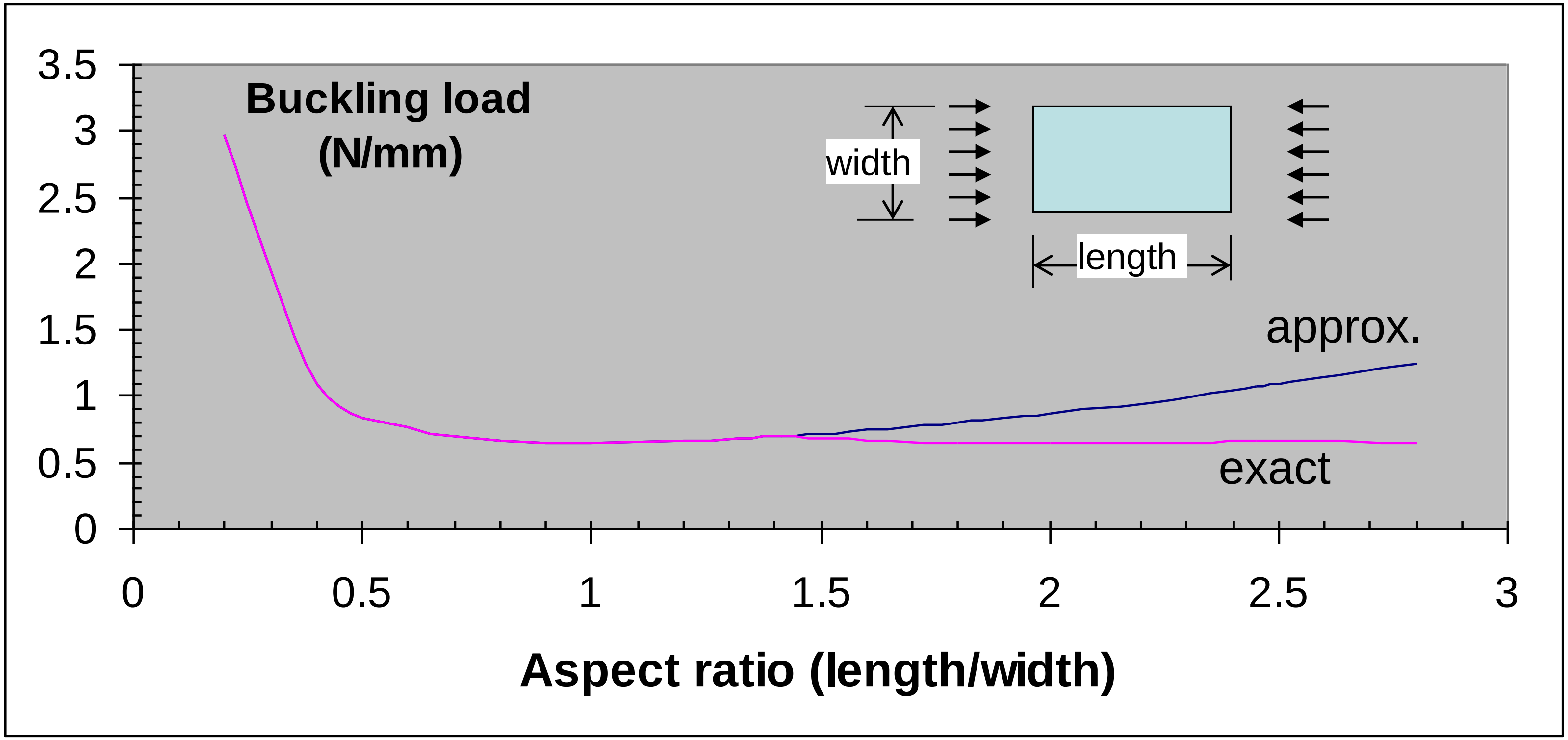
Special cases: Pure compression

- Nxy=0 => k=0

$$N_o = \frac{\pi^2}{a^2}\left(D_{11} + 2(D_{12} + 2D_{66})\frac{a^2}{b^2} + D_{22}\frac{a^4}{b^4}\right)$$

comparing with exact solution found earlier, this expression is identical to the exact solution for m=1 (but not so accurate for m>1)

Special case: Comparison to exact solution



D_{11}	0.66	Nm
D_{12}	0.47	
D_{22}	0.66	
D_{66}	0.49	

Special cases: Pure Shear

- set k very large

$$N_o = \frac{\pi^2}{a^2}\frac{\left(D_{11} + 2(D_{12} + 2D_{66})\frac{a^2}{b^2} + D_{22}\frac{a^4}{b^4}\right)}{2 - \frac{8192}{81}\frac{a^2}{b^2\pi^4}k^2}\left[5 \pm \sqrt{9 + \frac{65536}{81}\frac{a^2}{\pi^4 b^2}k^2}\right]$$

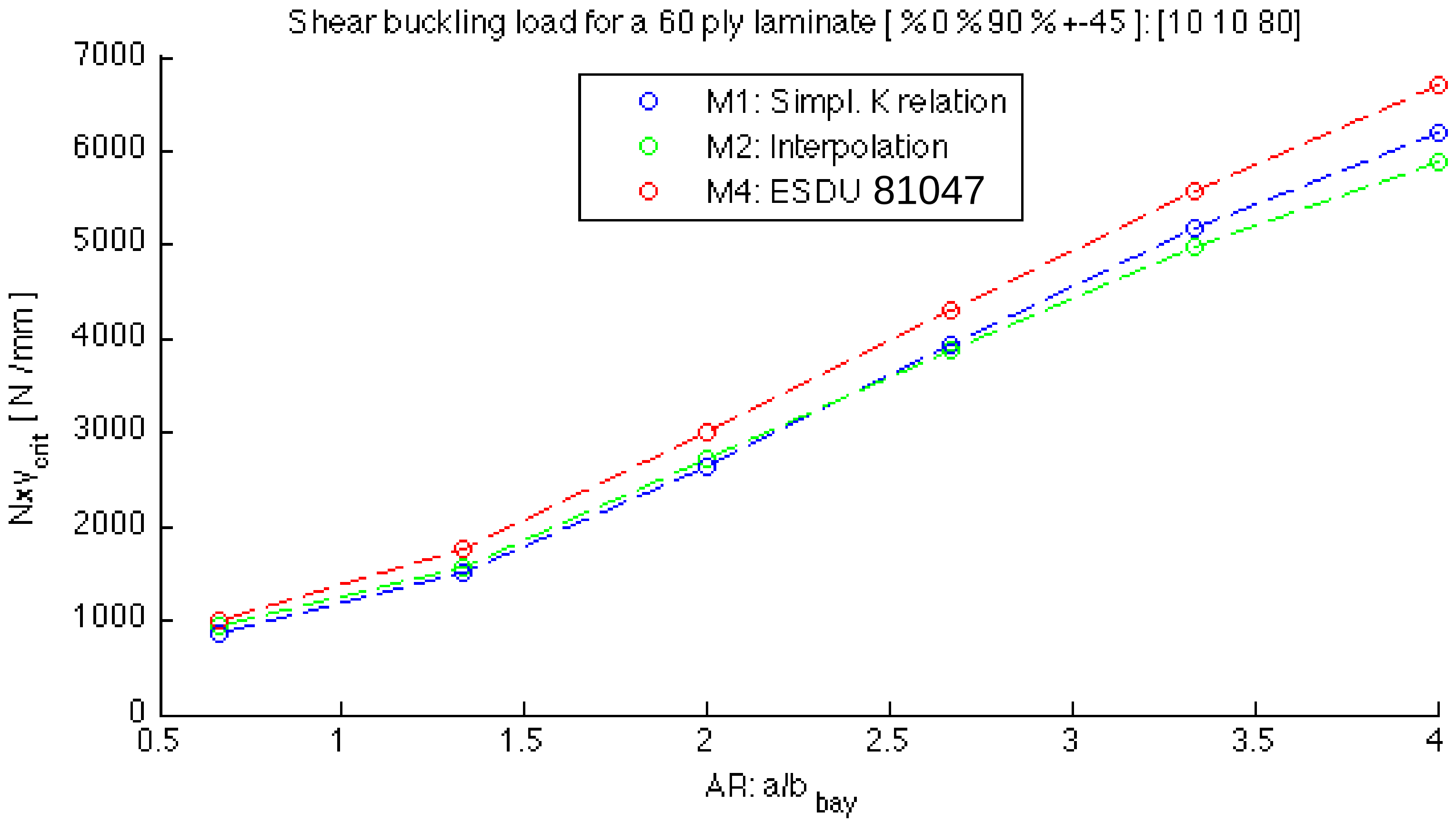
N_{xy}

$$N_o k = \pm \frac{\pi^2}{a^2}\frac{\left(D_{11} + 2(D_{12} + 2D_{66})\frac{a^2}{b^2} + D_{22}\frac{a^4}{b^4}\right)}{\frac{32}{9}\frac{a}{b\pi^2}}$$

$$N_{xycrit} = \frac{9\pi^4 b}{32a^3}\left(D_{11} + 2(D_{12} + 2D_{66})\frac{a^2}{b^2} + D_{22}\frac{a^4}{b^4}\right)$$

Typically, 27% higher than exact solution!

Shear buckling: Comparison of various methods[1]

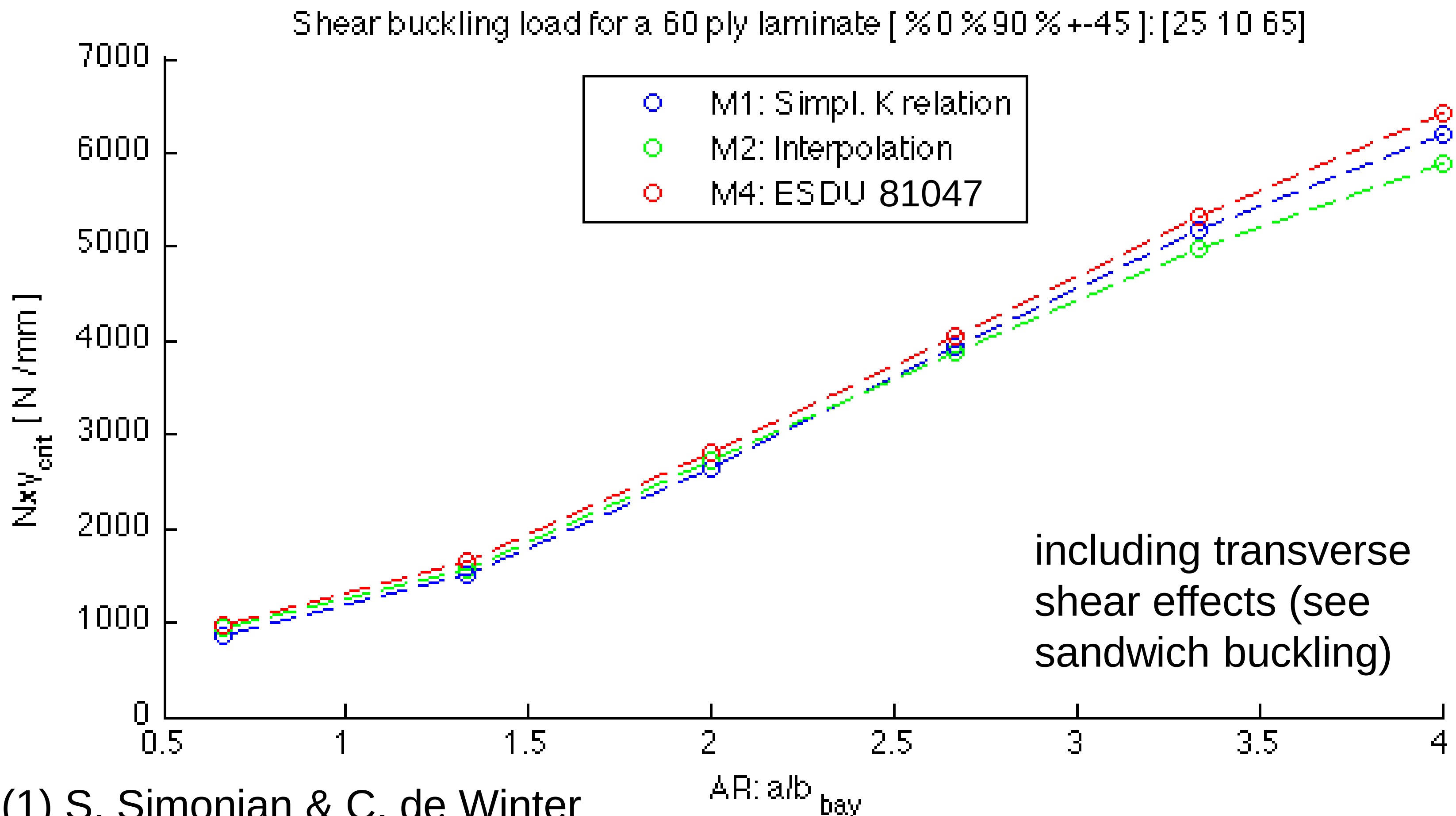


(1) S. Simonian & C. de Winter

K relation: equation just derived

Interpolation: see sandwich buckling under shear

Shear buckling: Comparison of various methods



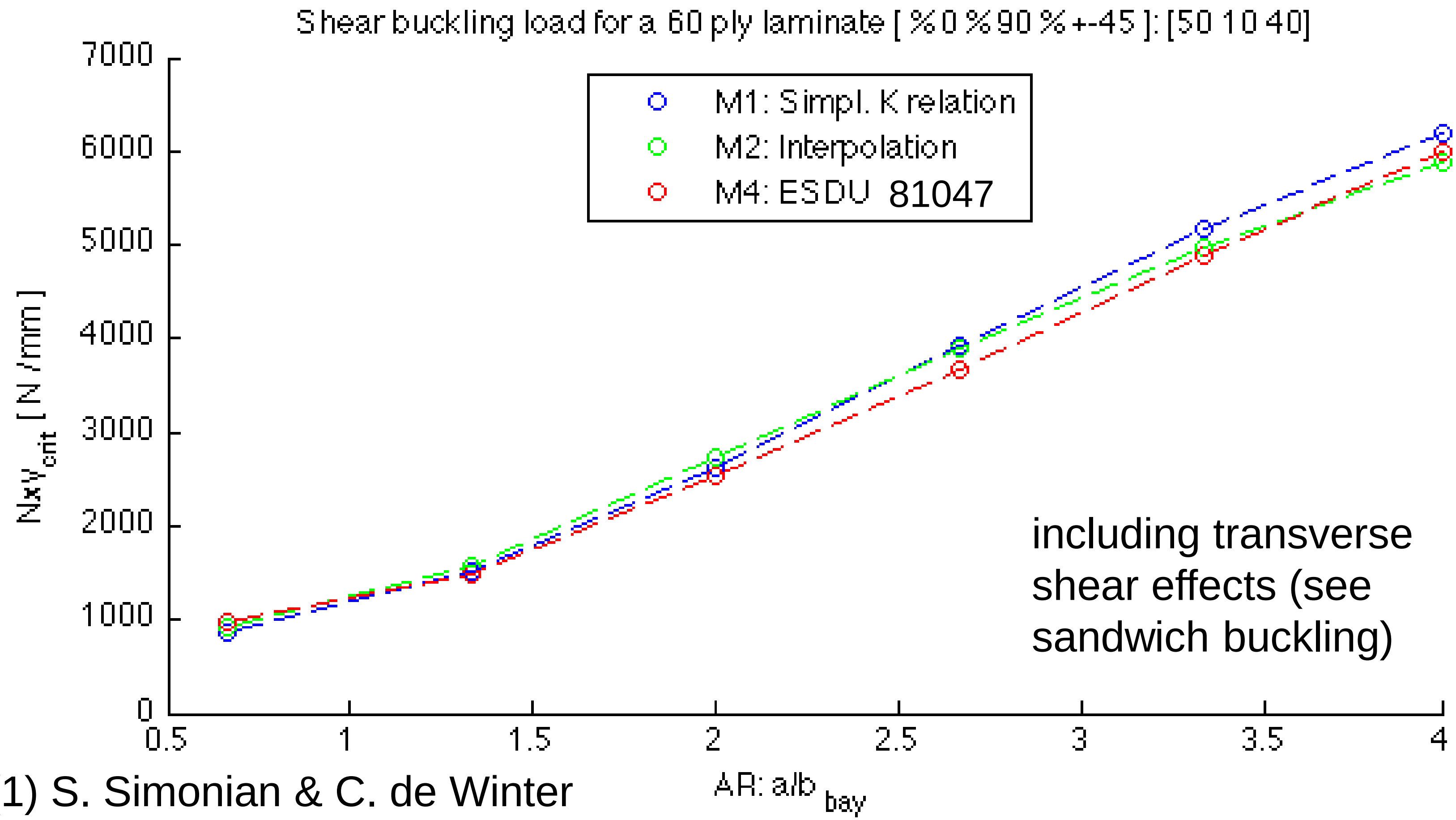
including transverse shear effects (see sandwich buckling)

(1) S. Simonian & C. de Winter

K relation: equation just derived

Interpolation: see sandwich buckling under shear

Shear buckling: Comparison of various methods



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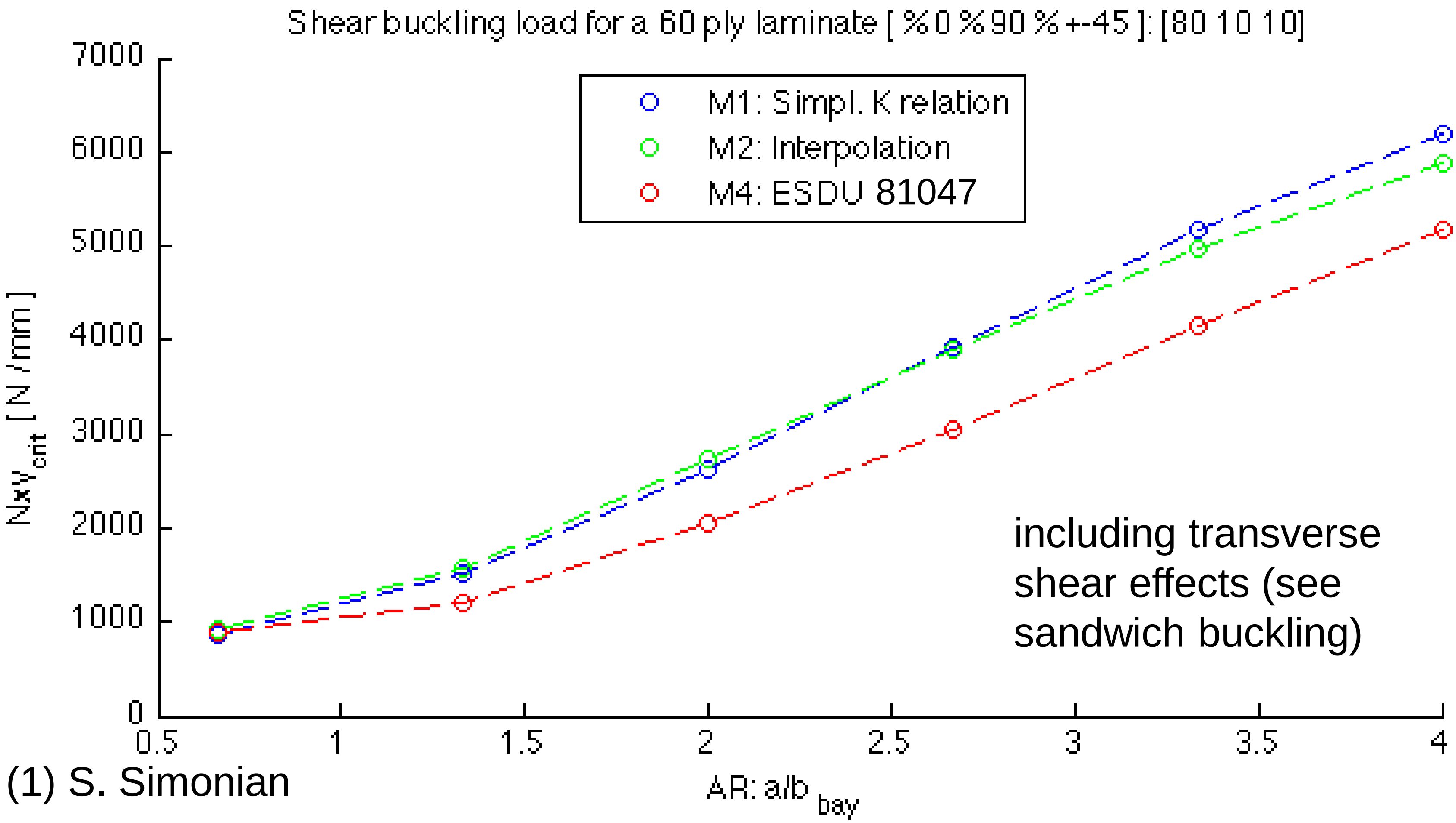


(1) S. Simonian

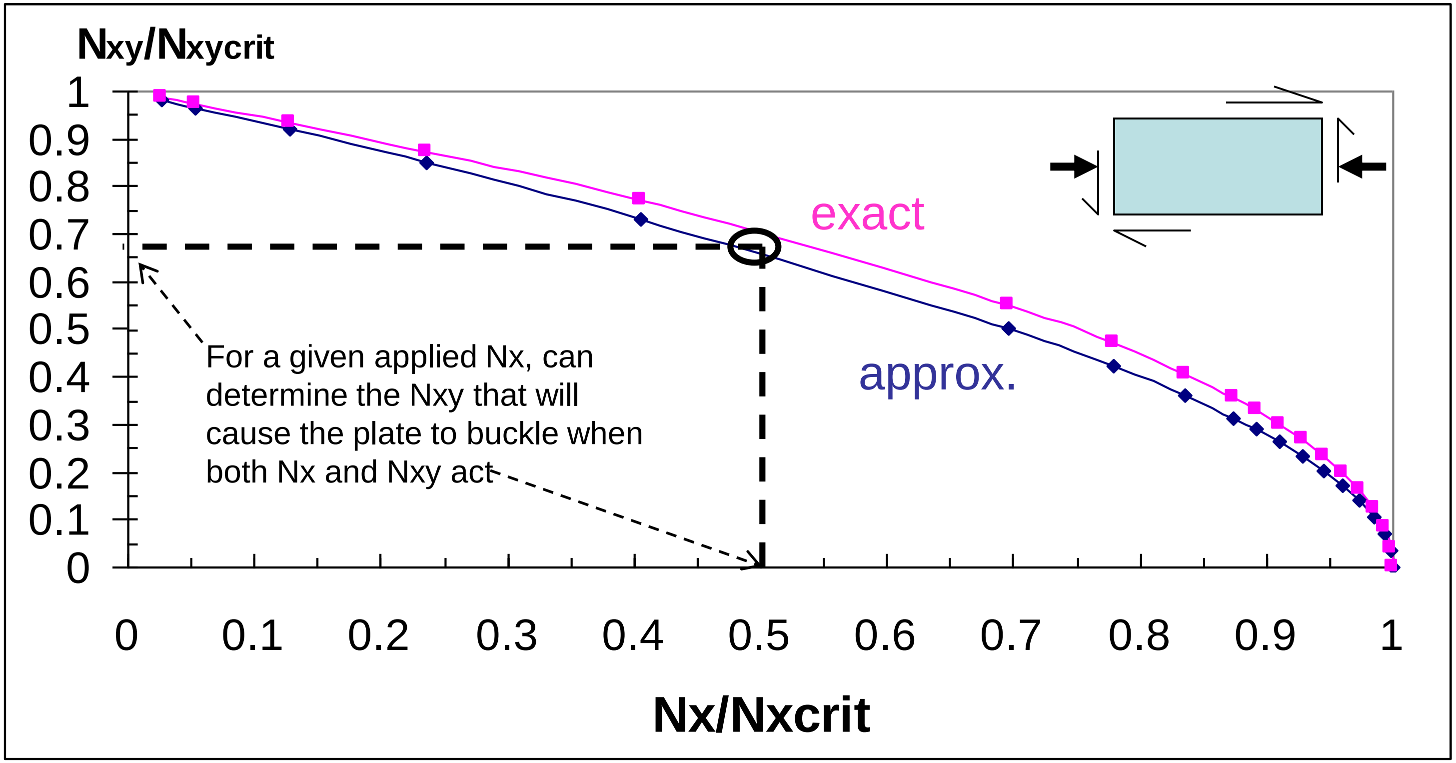
K relation: equation just derived

Interpolation: see sandwich buckling under shear

Interaction curve: Buckling under combined compression and shear

- Even though the present solution is approximate, it is expected to be quite accurate in providing the interaction curve when both compression and shear are applied

Buckling interaction curve: Combined compression and shear

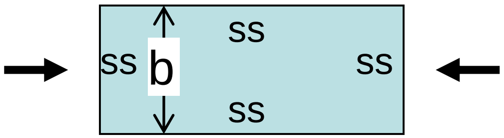


Here, N_{xcrit} and N_{xycrit} refer to the buckling when each load is applied <u>individually</u>

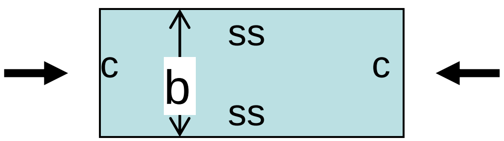
$$\frac{N_x}{N_{xcrit}} + \left(\frac{N_{xy}}{N_{xycrit}}\right)^2 = 1$$

"exact" interaction curve

Buckling under various loads and Boundary Conditions[1]



$$N_o = \frac{\pi^2\left[D_{11}m^4 + 2(D_{12}+2D_{66})m^2(AR)^2 + D_{22}(AR)^4\right]}{a^2 m^2}$$

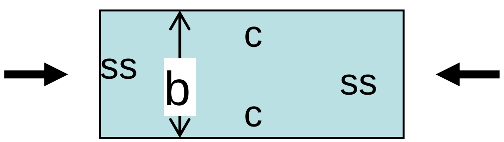


$$\lambda = \frac{a}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}$$

$$N_o = \frac{\pi^2}{b^2}\sqrt{D_{11}D_{22}}\,(K)$$

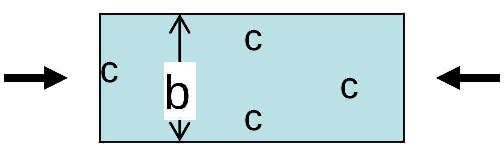
$$K = \frac{4}{\lambda^2} + \frac{2(D_{12}+2D_{66})}{\sqrt{D_{11}D_{22}}} + \frac{3}{4}\lambda^2 \qquad 0<\lambda<1.662$$

$$K = \frac{m^4+8m^2+1}{\lambda^2(m^2+1)} + \frac{2(D_{12}+2D_{66})}{\sqrt{D_{11}D_{22}}} + \frac{\lambda^2}{m^2+1} \qquad \lambda>1.662$$



$$\lambda = \frac{a}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}$$

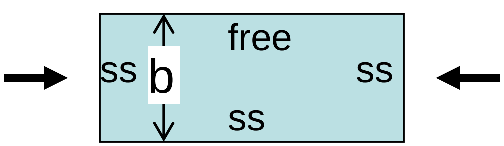
$$N_o = \frac{\pi^2}{b^2}\sqrt{D_{11}D_{22}}\,(K) \qquad K = \frac{m^2}{\lambda^2} + \frac{2(D_{12}+2D_{66})}{\sqrt{D_{11}D_{22}}} + \frac{16}{3}\frac{\lambda^2}{m^2}$$



$$\lambda = \frac{a}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}$$

$$N_o = \frac{\pi^2}{b^2}\sqrt{D_{11}D_{22}}\,(K) \quad K = \frac{4}{\lambda^2} + \frac{8(D_{12}+2D_{66})}{3\sqrt{D_{11}D_{22}}} + 4\lambda^2 \qquad 0<\lambda<1.094$$

$$K = \frac{m^4+8m^2+1}{\lambda^2(m^2+1)} + \frac{2(D_{12}+2D_{66})}{\sqrt{D_{11}D_{22}}} + \frac{\lambda^2}{m^2+1} \qquad \lambda>1.094$$

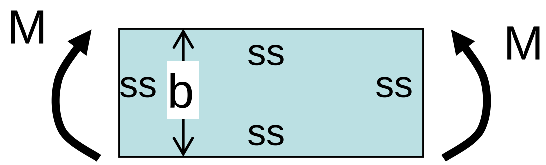


$$\lambda = \frac{a}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}$$

$$N_o = \frac{\pi^2}{b^2}\sqrt{D_{11}D_{22}}\,(K) \qquad K = \frac{12}{\pi^2}\frac{D_{66}}{\sqrt{D_{11}D_{22}}} + \frac{1}{\lambda^2}$$

(1) NASA/DoD Adv Composites Design Guide, vol II, 1983

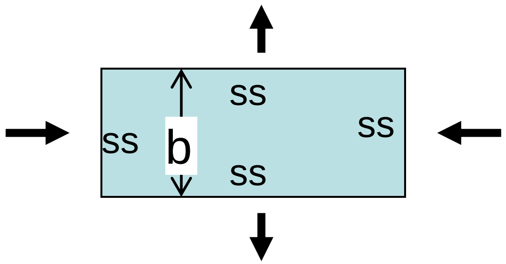
Buckling under various loads and boundary conditions[1]



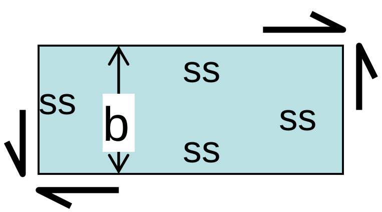
$$\lambda = \frac{a}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}$$

$$M_o = \frac{\pi^2}{b^2}\sqrt{D_{11}D_{22}}\,(K)$$

$$K = 0.047\pi^2 b^2 \sqrt{\left(\frac{m^2}{\lambda^2} + \frac{2(D_{12}+2D_{66})}{\sqrt{D_{11}D_{22}}} + \frac{\lambda^2}{m^2}\right)\left(\frac{m^2}{\lambda^2} + \frac{8(D_{12}+2D_{66})}{\sqrt{D_{11}D_{22}}} + 16\frac{\lambda^2}{m^2}\right)}$$



$$N_o = \frac{\pi^2\left[D_{11}m^4 + 2(D_{12}+2D_{66})m^2n^2(AR)^2 + D_{22}n^4(AR)^4\right]}{a^2(m^2 + kn^2(AR)^2)}$$



$$\lambda = \frac{a}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}$$

a=b;

$0<\beta\leq 1$

$$N_{xycrit} = \frac{4}{b^2}\left(D_{11}D_{22}{}^3\right)^{1/4}(K)$$

$$K = 8.2 + 5\frac{(D_{12}+2D_{66})}{\sqrt{D_{11}D_{22}}}\frac{1}{10^{(\frac{A}{\beta}+B\beta)}}$$

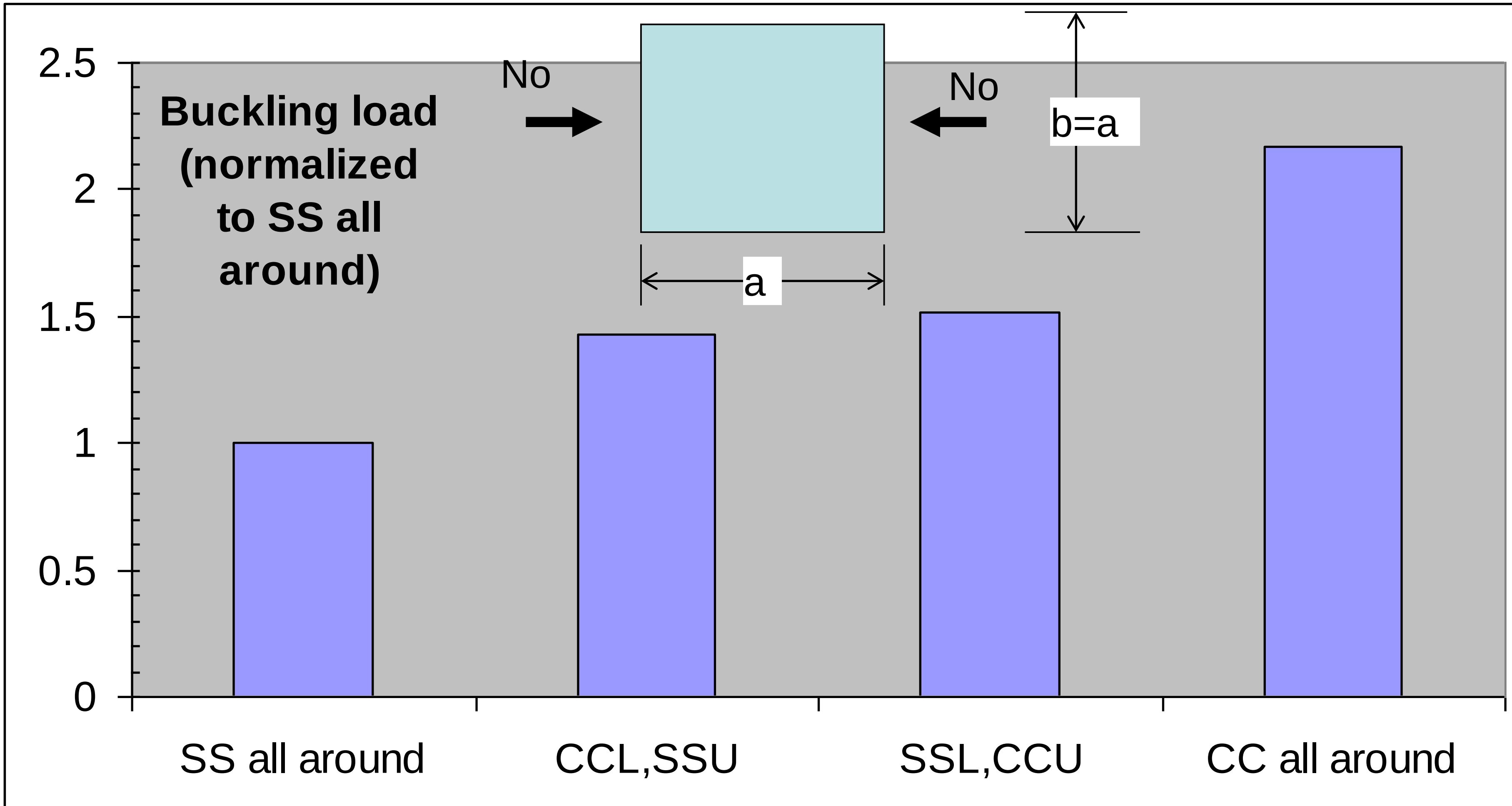
$$\beta = \left(\frac{D_{11}}{D_{22}}\right)^{1/4}$$

$$A = -0.27 + 0.185\frac{(D_{12}+2D_{66})}{\sqrt{D_{11}D_{22}}}$$

$$B = 0.82 + 0.46\frac{(D_{12}+2D_{66})}{\sqrt{D_{11}D_{22}}} - 0.2\left(\frac{(D_{12}+2D_{66})}{\sqrt{D_{11}D_{22}}}\right)^2$$

(1) NASA/DoD Adv Composites Design Guide, vol II, 1983

Effect of BC's on buckling load of a square plate under compression



Biggest difference is less than 2.5 to 1 (compare to beams with 4 to 1 ratio)