Time domain models
Course: Wb2301

Lecture 6

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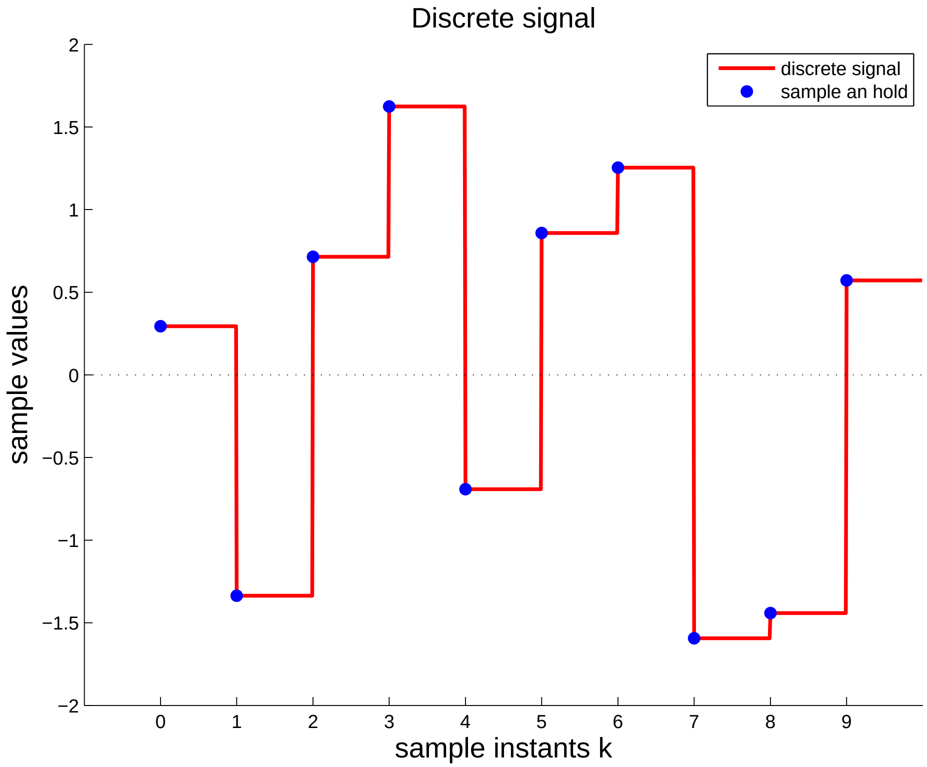
Contents

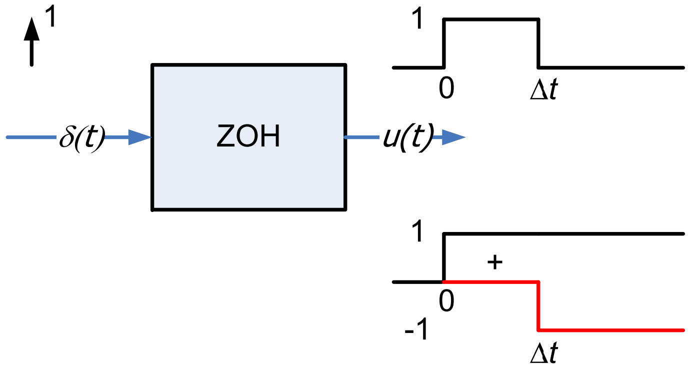
- Distortion from Sample and hold circuits
- The Z-transformation
- Discrete time-domain identification
- Model structures and parameterization
- Open and closed loop identification using discrete models

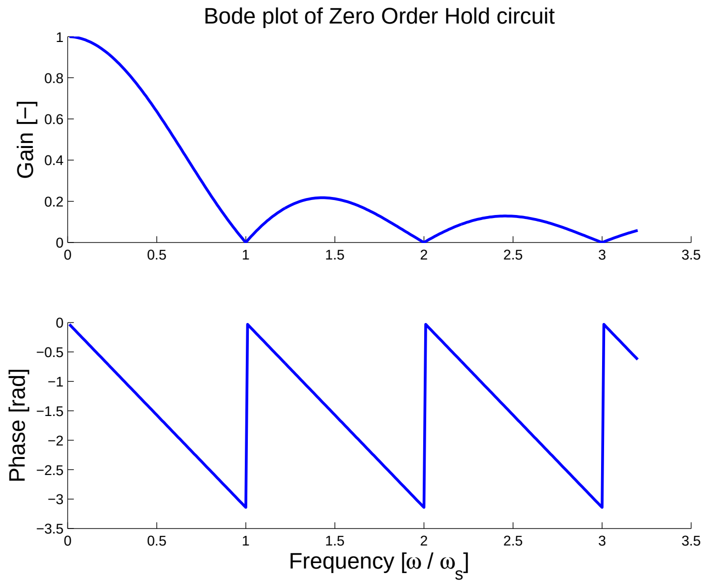
Signal Reconstruction



Zero order hold (ZOH)
Sample time $\Delta t = 1s$

ZOH FRF



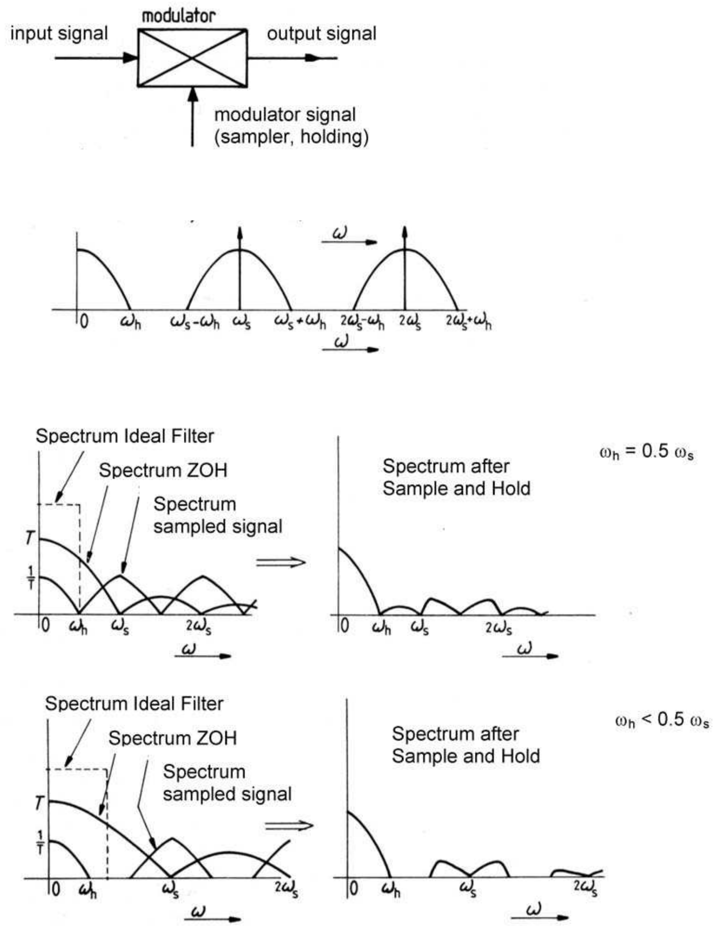


FRF = $\mathscr{L}$(IRF):

$$H_{zoh}(s) = \frac{1}{s} - \frac{e^{-s\Delta t}}{s}$$

$$\begin{aligned} H_{zoh}(j\omega) &= \frac{e^{-\frac{1}{2}j\omega\Delta t}\left(e^{\frac{1}{2}j\omega\Delta t} - e^{-\frac{1}{2}j\omega\Delta t}\right)}{j\omega} \\ &= \frac{e^{-\frac{1}{2}j\omega\Delta t}\cdot 2j\sin\frac{1}{2}\omega\Delta t}{j\omega} \\ &= \frac{\Delta t\sin\frac{1}{2}\omega\Delta t\cdot e^{-\frac{1}{2}j\omega\Delta t}}{\frac{1}{2}\omega\Delta t} \end{aligned}$$

Signal Distortion



- Signal distortion due to sampling and hold circuit
- Reduce distortion by increasing sampling frequency
- No signal transfer at the Nyquist frequency ($\omega_h = \frac{1}{2}\omega_s$ and sample frequency (ω_s)

Z-transformation

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where n is an integer and z is, in general, a complex number:

$$\begin{aligned} z &= Ae^{j\phi} \\ &= A(cos\phi + jsin\phi) \end{aligned}$$

where A is the magnitude, and ϕ is the angle of z

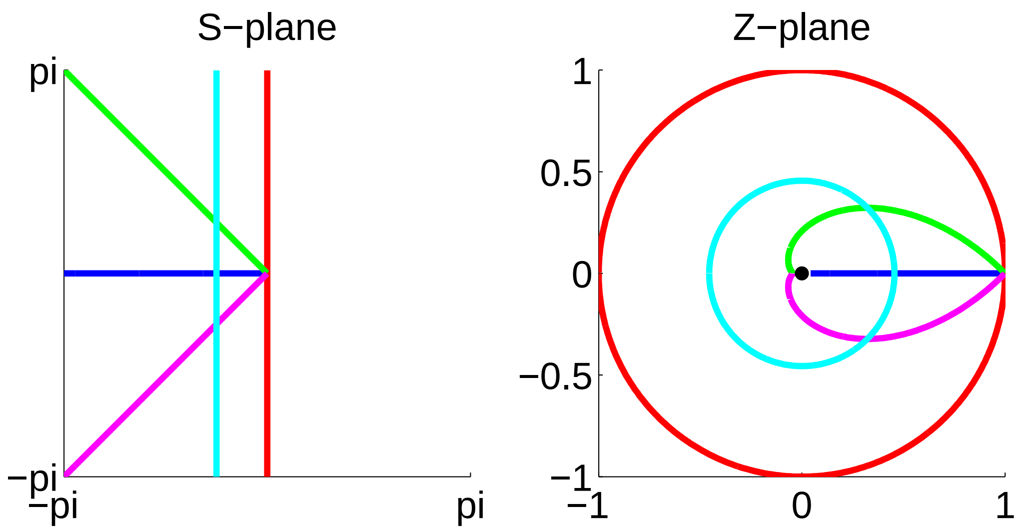
Z-transformation

Example:

$$
\begin{aligned}
[x(1)\ x(2)\ x(3)\ ...\ x(N)] &= [1\ 2\ 3\ 0\ 0\ ...\ 0] \\
X(z) &= 1z^0 + 2z^{-1} + 3z^{-2}
\end{aligned}
$$

Continuous to discrete poles-zeros

$$
\begin{aligned}
z &= e^{s\Delta t} \\
&= e^{Re(s)\Delta t} e^{Im(s)\Delta t}
\end{aligned}
$$



Z-Laplace Mappings

Inverse mapping

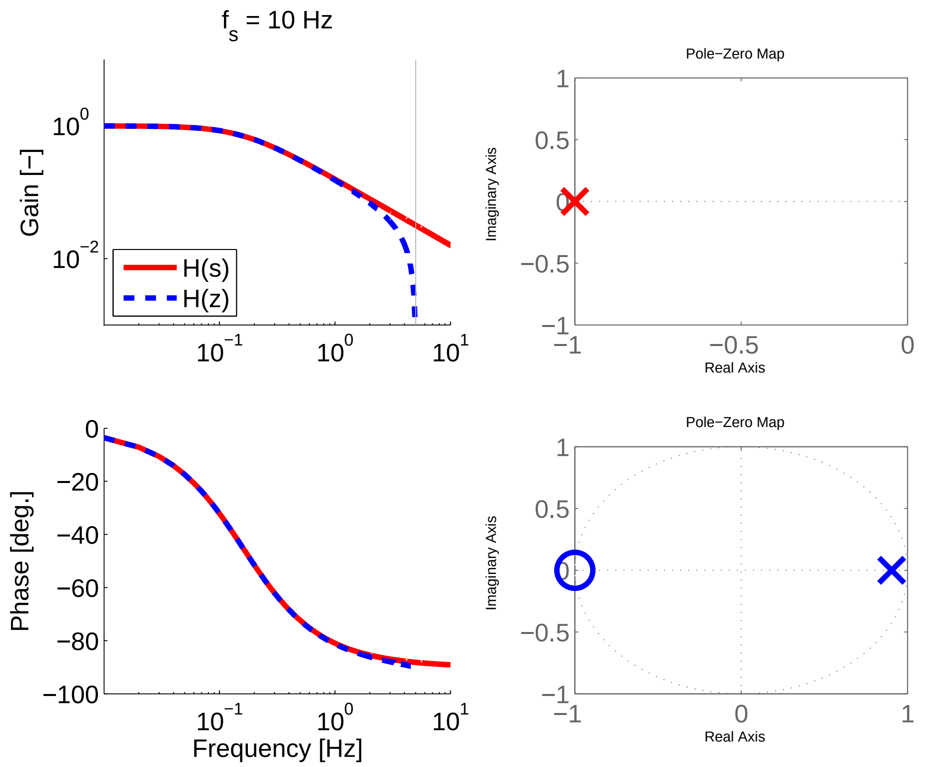
$$s = \frac{1}{\Delta t}\ln(z)$$

Using the bilinear (Tustin) approximation (1st order Taylor coeff.):

$$\begin{aligned} s &= \frac{2}{\Delta t}\frac{z-1}{z+1} = \frac{2}{\Delta t}\frac{1-z^{-1}}{1+z^{-1}} \\ z &= \frac{2+s\Delta t}{s-s\Delta t} \end{aligned}$$

Example

$$H(s) = \frac{1}{s+1} \qquad H(z)_{tustin} = \frac{\Delta t}{2+\Delta t} \frac{z+1}{z - \frac{2-\Delta t}{2+\Delta t}}$$



Time domain identification

1. Digital acquisition: signals are represented by discrete numbers (sampling)
2. Systems are described by polynomials in the z-domain
   - Deterministic and stochastic parts can be separated (noise models)
   - Appropriate for nonlinear identification
   - Disadvantage: a priori knowledge required

Time domain identification

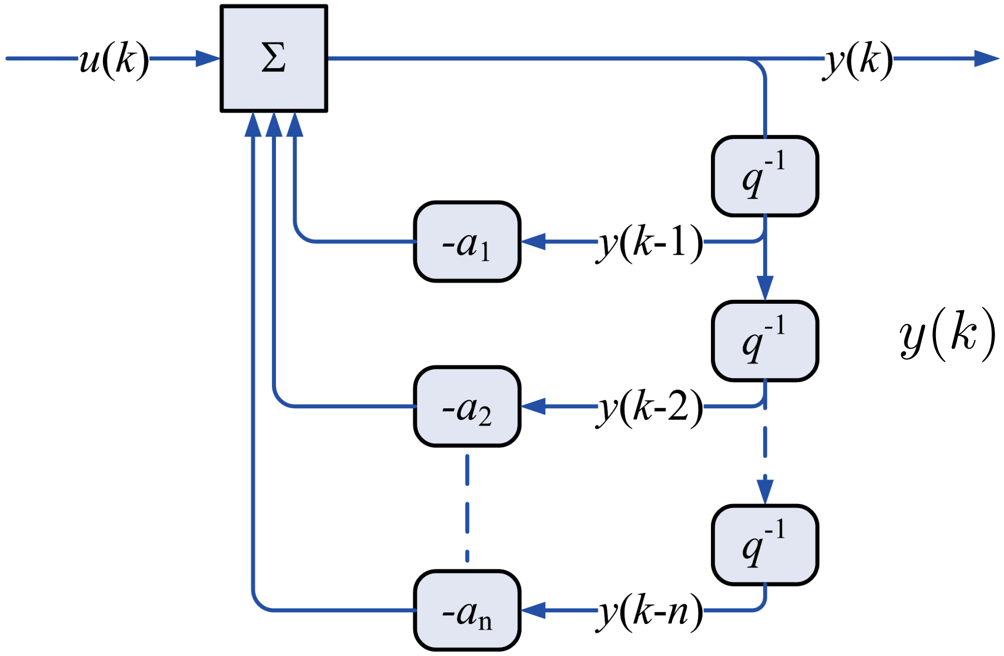
- Purpose

  Capture system properties by a limited number of parameters ($n < N$) These n parameters (indirectly!) represent system dynamics, i.e. physical properties

- Approach

  Discriminate input related response from noise in output

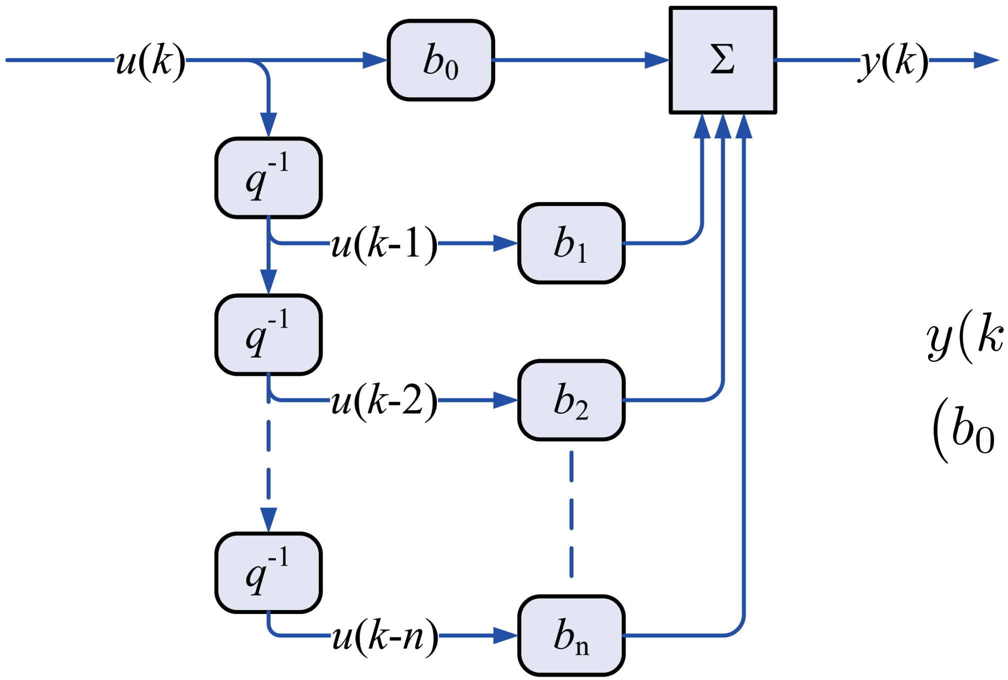
Auto-regressive (AR) filter



$$y(k) = \frac{1}{a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n}} u(k)$$

Describes many physical processes due to feed-back structure

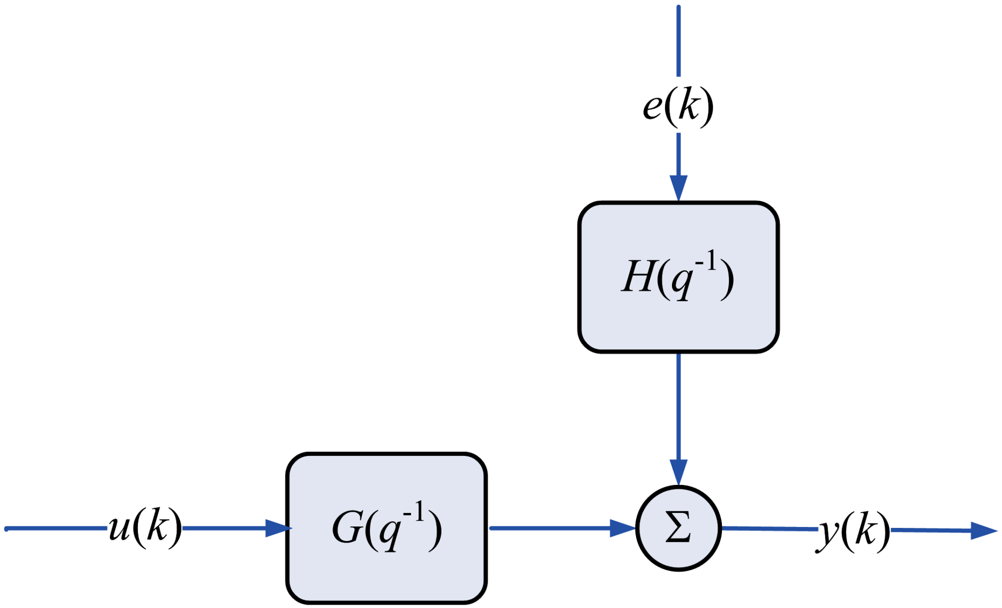
Moving Average (MA) filter



$$y(k) =$$
$$\left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n z^{-n}\right) u(k)$$

Finite time integration structure

General Model



$$y(k) = G(z^{-1})u(k) + H(z^{-1})e(k)$$

- white noise $e(k)$
- goal: estimate $G(z^{-1})$ from $y(k), u(k)$

System Identification

- Experimental system data
- Choice of model structure
- Optimization criterion
- Validation

Finding the best model

System:

$$y(k) = G_0(z^{-1})u(k) + H_0(z^{-1})e(k)$$

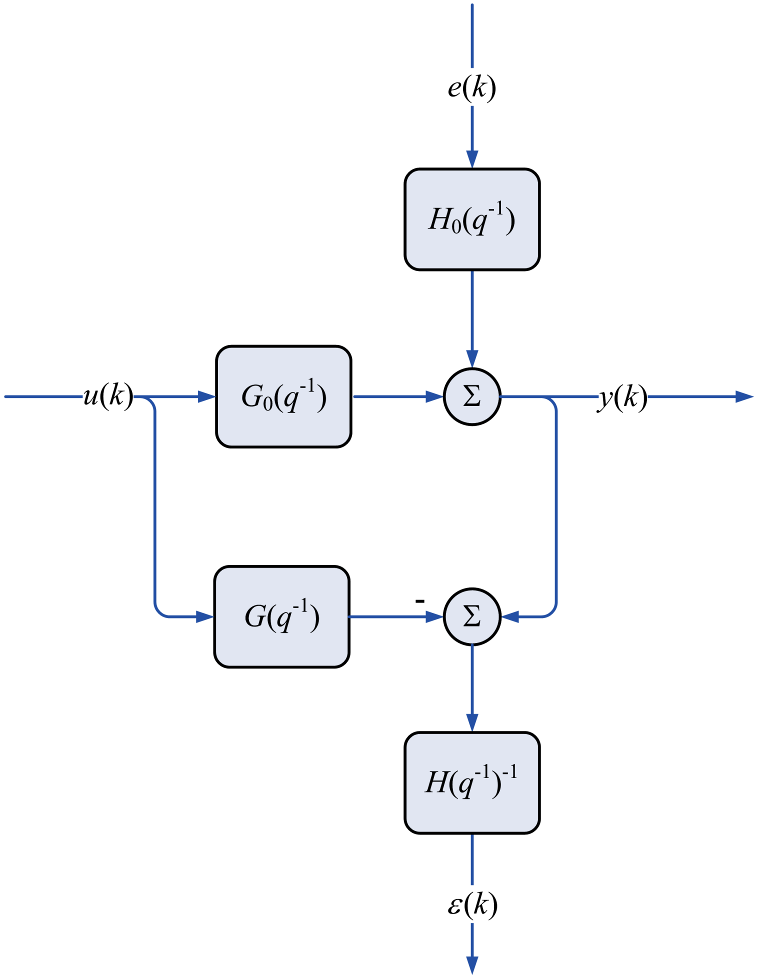
Assume a model (G, H, ϵ) can be found such that:

$$y(k) = G(z^{-1})u(k) + H(z^{-1})\epsilon(k)$$

Then the model error is:

$$\epsilon(k) = H^{-1}(z^{-1})\left[y(k) - G(z^{-1})u(k)\right]$$

Obtaining the model



System: $G_0(z^{-1}), H_0(z^{-1})$

Model: $G(z^{-1}), H(z^{-1})$

$$\epsilon(k) \quad = \quad H^{-1}(z^{-1})\left[y(k) - G(z^{-1})u(k)\right]$$

Optimal fit :

If $G(z^{-1}) \approx G_0(z^{-1})$

and $H(z^{-1}) \approx H_0(z^{-1})$

then $\epsilon(k) \approx e(k)$

Least squares (LS) Criterion

$$V(\theta)_N = min \sum_{k=1}^{N} \epsilon(k,\theta)^2$$

V cost function

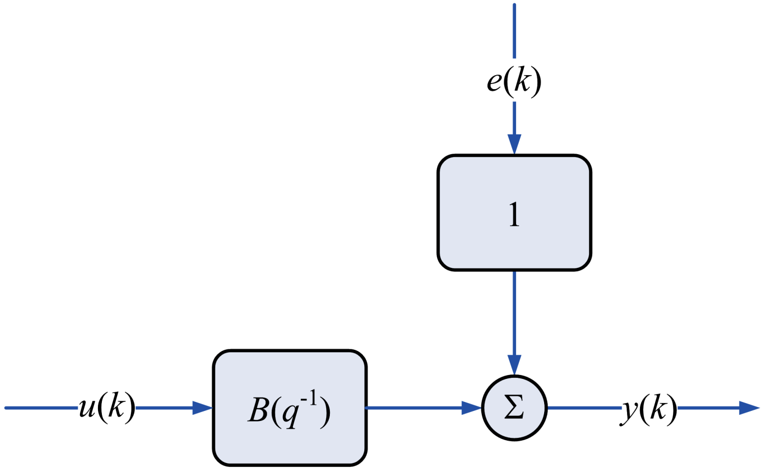
θ model parameter vector

ϵ model error

N number of data points

For the optimal solution: $\epsilon \rightarrow e$

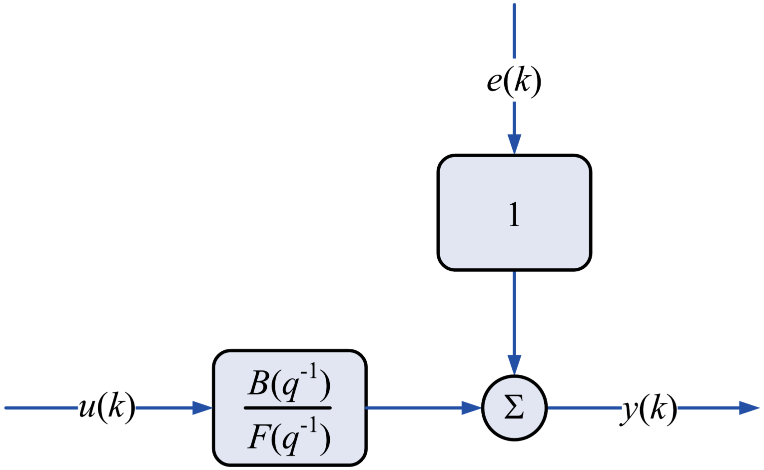
Finite Impulse Response (FIR) structure



$$
\begin{aligned}
y(k) &= B(z^{-1})u(k) + e(k) \\
e(k) &= y(k) - B(z^{-1})u(k)
\end{aligned}
$$

An FIR model is linear in its parameters.

Output Error (OE) structure

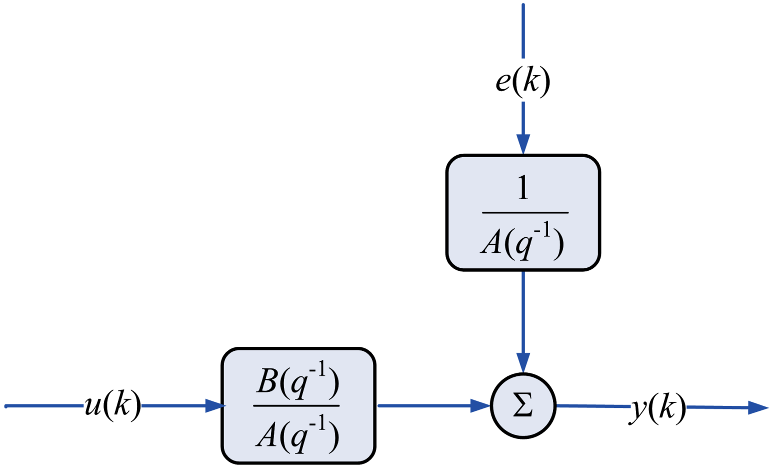


$$y(k) = \frac{B(z^{-1})}{F(z^{-1})}u(k) + e(k)$$

$$e(k) = y(k) - \frac{B(z^{-1})}{F(z^{-1})}u(k)$$

An OE model is <u>nonlinear</u> in its parameters.

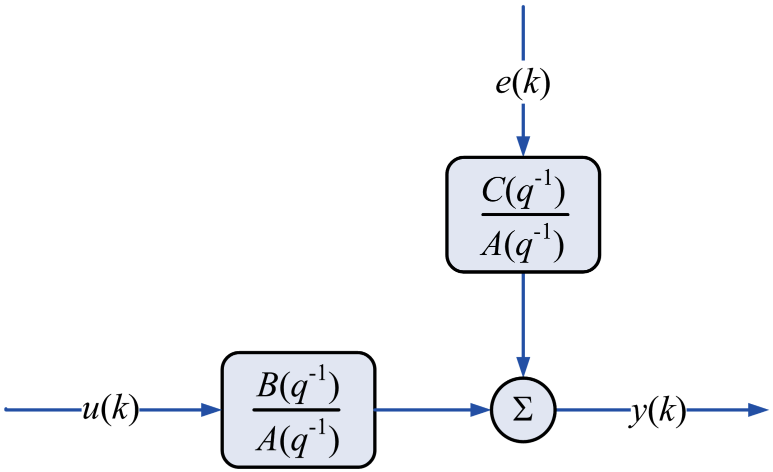
ARX structure



$$
\begin{aligned}
y(k) &= \frac{B(z^{-1})}{A(z^{-1})}u(k) + \frac{1}{A(z^{-1})}e(k) \\
e(k) &= A(z^{-1})y(k) - B(z^{-1})u(k)
\end{aligned}
$$

An ARX model is linear in its parameters.

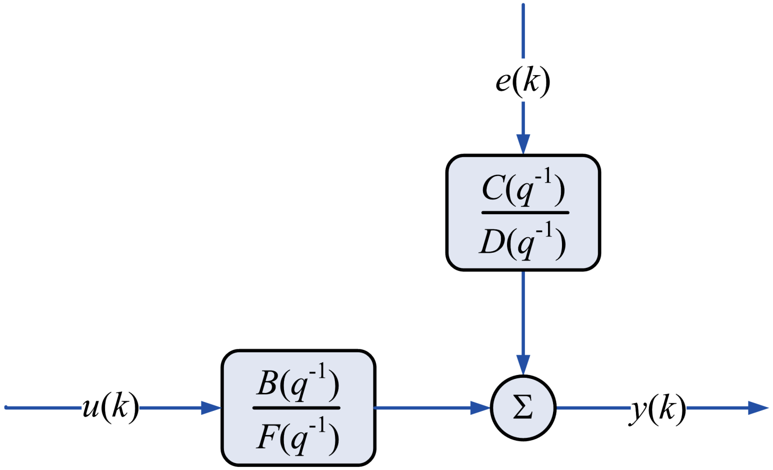
ARMAX structure



$$
\begin{aligned}
y(k) &= \frac{B(z^{-1})}{A(z^{-1})}u(k) + \frac{C(z^{-1})}{A(z^{-1})}e(k) \\
e(k) &= \frac{A(z^{-1})}{C(z^{-1})}y(k) - \frac{B(z^{-1})}{C(z^{-1})}u(k)
\end{aligned}
$$

An ARMAX model is <u>nonlinear</u> in its parameters.

Box-Jenkins (BJ) structure



$$y(k) = \frac{B(z^{-1})}{F(z^{-1})}u(k) + \frac{C(z^{-1})}{D(z^{-1})}e(k)$$

$$e(k) = \frac{D(z^{-1})}{C(z^{-1})}y(k) - \frac{D(z^{-1})B(z^{-1})}{C(z^{-1})F(z^{-1})}u(k)$$

An BJ model is nonlinear in its parameters.

Optimizing the criterion (ARX)

$$\epsilon(k) = A(z^{-1})y(k) - B(z^{-1})u(k)$$

with n_a, n_b are the model orders according to:

$$\begin{aligned} na: \quad & A(z^{-1}) = 1 + a_1 z^{-1} + \ldots + a_{na} z^{-na} \\ nb: \quad & B(z^{-1}) = b_1 + b_2 z^{-1} \ldots + b_{nb} z^{-nb+1} \end{aligned}$$

Structured regression format:

$$\begin{aligned} \epsilon(k) &= y(k) + [A(z^{-1}) - 1]y(k) - B(z^{-1})u(k) \\ &= y(k) - \phi(z^{-1})\theta \end{aligned}$$

where

$$\begin{aligned} \theta &= [b_1 \;\; b_2 \; ... \; b_n \;\; a_1 \;\; a_2 \; ... \; a_n]^T \\ \phi &= [U \;\; -Y] \end{aligned}$$

Optimizing the criterion (ARX)

$$
\begin{aligned}
V(\theta)_N &= min\sum_{k=1}^{N}\epsilon(k,\theta)^T\epsilon(k,\theta) = \epsilon(k)^T\epsilon(k) \\
\frac{\delta V(\theta)}{\delta\theta} &= [y(k)-\phi\theta]^T\phi = 0
\end{aligned}
$$

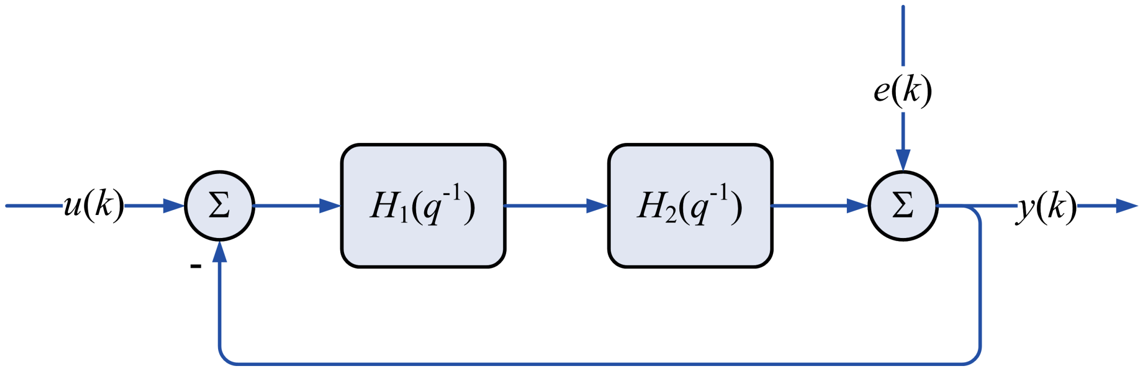
$$
\theta = y(k)^T\phi[\phi^T\phi]^{-1}
$$

Choice of model structure

Use system knowledge

- Existence of feedback loops
- Noise sources

Closed loop system



Goal: estimate $\frac{y(k)}{u(k)}$

Take:

$$H_1(z^{-1}) = \frac{N_1}{D_1}$$

$$H_2(z^{-1}) = \frac{N_2}{D_2}$$

$$\begin{aligned} y &= \frac{H_1 H_2}{1 + H_1 H_2} u + \frac{1}{1 + H_1 H_2} e \\ &= \frac{N_1 N_2}{D_1 D_2 + N_1 N_2} u + \frac{D_1 D_2}{D_1 D_2 + N_1 N_2} e \\ &= \frac{B}{A} u + \frac{C}{A} e \end{aligned}$$

This system has an ARMAX structure.

Analysis

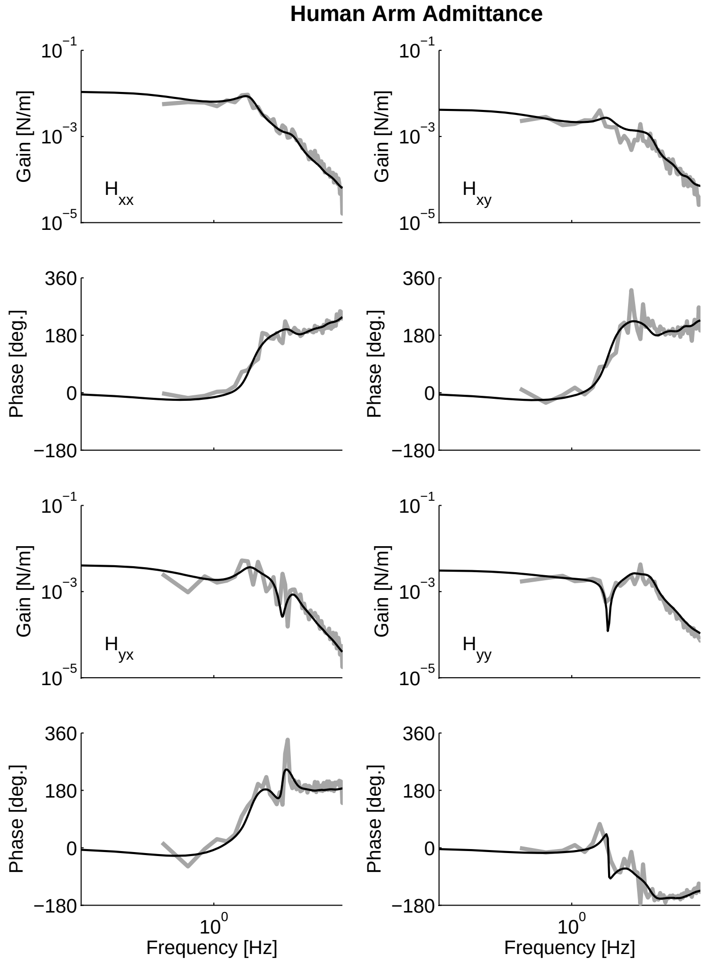
Frequency response:

$$z^{-n} = e^{-ns\Delta t} \text{ with } s = j2\pi f \quad (f = [0 \ldots \frac{f_s}{2}])$$

Impulse response:

$$u(k) = m \text{ with } \begin{array}{l} m = 1 \text{ for } k = 0 \\ m = 0 \text{ for } k > 0 \end{array}$$

Example

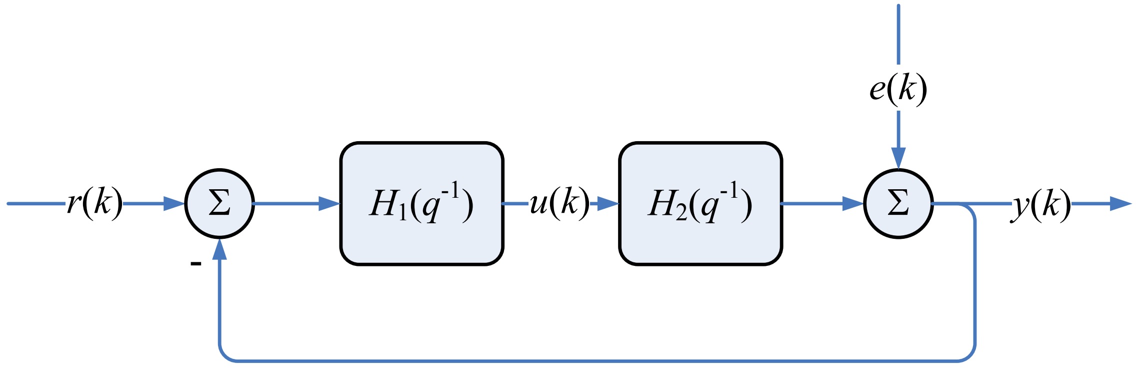


2DOF arm admittance:

spectral (grey) ARX (black)

Closed loop algorithms

Goal: estimate H_2



1. Two stage method
2. Coprime factorization method

Two stage

System transferfunctions:

$$
\begin{aligned}
u(k) &= \frac{H_1}{1+H_1H_2}r(k) + \frac{H_1}{1+H_1H_2}e(k) \\
y(k) &= \frac{H_1H_2}{1+H_1H_2}r(k) + \frac{1}{1+H_1H_2}e(k)
\end{aligned}
$$

Two stage:

1. Open loop from $r(k)$ to $u(k)$
2. Simulate a 'noise free' input $u^*(k)$
3. Open loop estimate from $u^*(k)$ to $y(k)$ gives the estimated system H_2

Drawback: only for stable (sub)systems because of the simulation step

Coprime factorization

System transferfunctions:

$$
\begin{aligned}
u(k) &= \frac{H_1}{1+H_1H_2}r(k) + \frac{H_1}{1+H_1H_2}e(k) \\
y(k) &= \frac{H_1H_2}{1+H_1H_2}r(k) + \frac{1}{1+H_1H_2}e(k)
\end{aligned}
$$

Coprime factorization:

1. Open loop from $r(k)$ to $u(k)$ gives $\frac{H_1}{1+H_1H_2}$
2. Open loop from $r(k)$ to $y(k)$ gives $\frac{H_1H_2}{1+H_1H_2}$
3. then, $\frac{y(k)}{r(k)}\frac{r(k)}{u(k)}$ gives H_2

Drawback: high order because poles and zeros do not cancel out

Summary

- Time domain models use only a few parameters (w.r.t. FRFs)
- Model structure required and must be selected a priori
- Optimum order must be selected
- Parameterization by minimizing an error criterion
- Direct simulation/validation