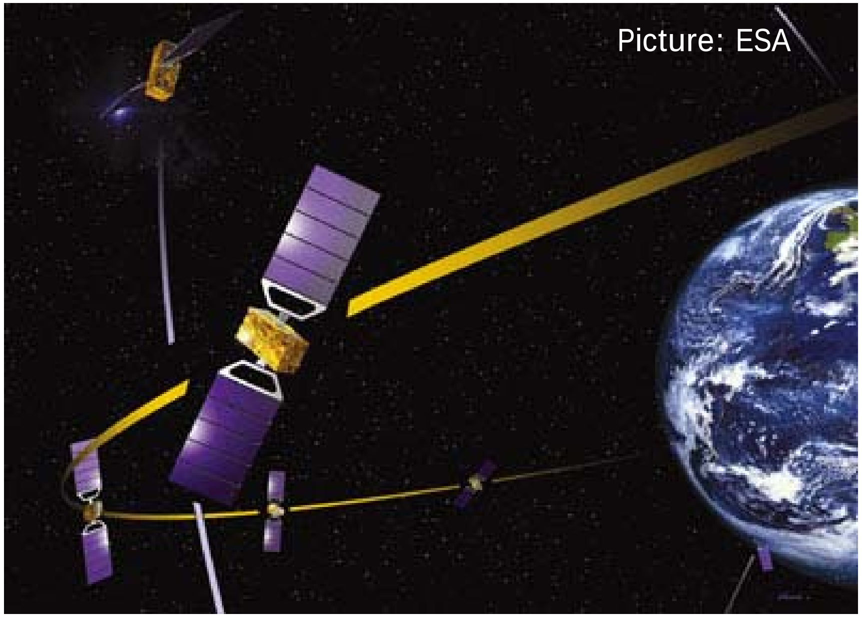
Satellite Navigation
error sources and position estimation



AE4E08

Sandra Verhagen

Course 2010 – 2011, lecture 6

TU Delft

Today's topics

- Recap: GPS measurements and error sources
- Signal propagation errors: troposphere
- Multipath
- Position estimation

- Book: Sections 5.3 – 5.7, 6.1.1

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Recap: error sources

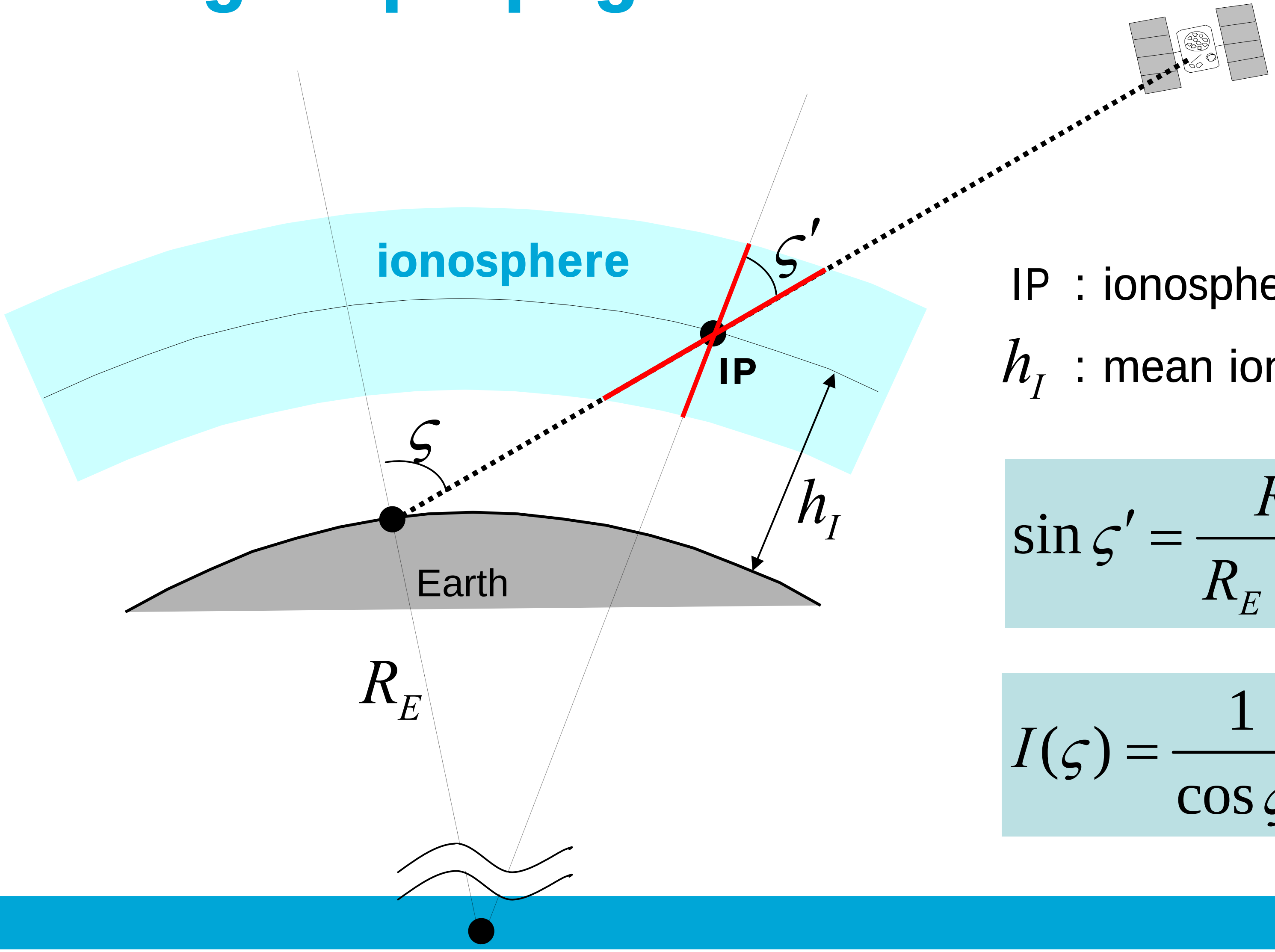
- satellite:
  - orbit
  - clock
  - instrumental delays
- signal path
  - **ionosphere**
  - **troposphere**
  - **multipath**
- receiver
  - clock
  - instrumental delays
- other
  - spoofing
  - interference

Recap: Code and Carrier Phase measurements

$$\rho_{Li} = r + \frac{f_1^2}{f_i^2} I_{L1} + T + c\left[\delta t_u - \delta t^s\right] + \varepsilon_{\rho_{Li}}$$

$$\Phi_{Li} = r - \frac{f_1^2}{f_i^2} I_{L1} + T + c\left[\delta t_u - \delta t^s\right] + \lambda A_{Li} + \varepsilon_{\Phi_{Li}}$$

Signal propagation errors: ionosphere



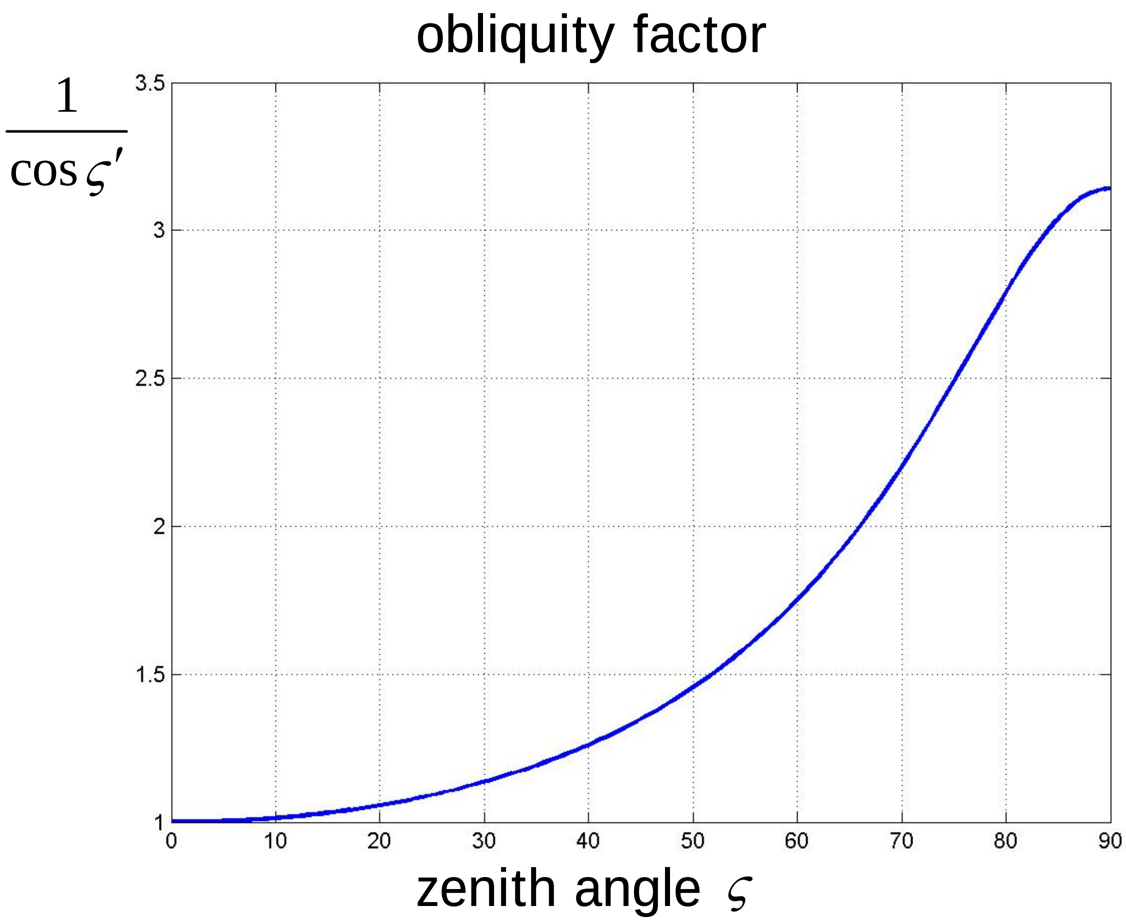
IP : ionospheric pierce point

h_I : mean ionosphere height

$$\sin\varsigma' = \frac{R_E}{R_E + h_I}\sin\varsigma$$

$$I(\varsigma) = \frac{1}{\cos\varsigma'} I_z$$

Signal propagation errors: ionosphere



Signal propagation errors: ionosphere

zenith delay mid-latitudes:

- 1-3 m at night
- 5-15 m mid-afternoon

peak solar cycle near equator:

- max. ~36 m

Signal propagation errors: ionosphere

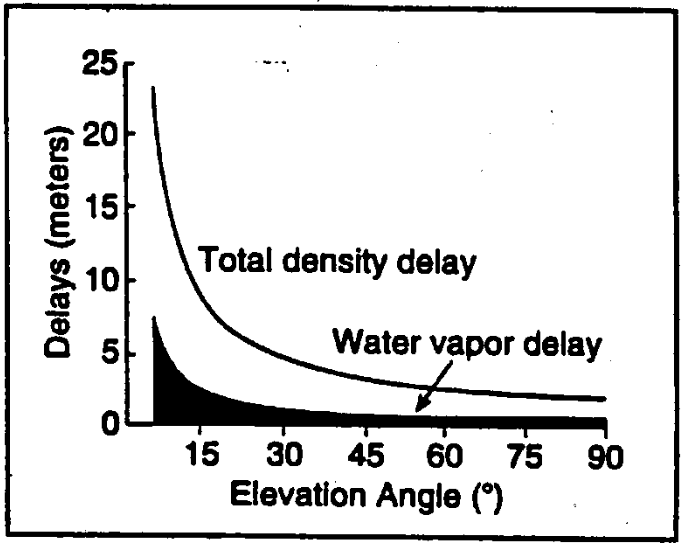
How to deal with ionosphere?

- apply ionosphere-free combination (dual-frequency receiver required)
- apply model (reduction 50 – 70%)
- relative positioning (later this course)

Signal propagation errors: troposphere

- 9 km (poles) – 16 km (equator)
- Dry gases and water vapor
- Recall: non-dispersive, i.e. refraction does not depend on frequency
- Propagation speed lower than in free space:
  apparent range is longer (~2.5 – 25 m)
- Same phase and group velocities

$$T_{\rho_{L1}} = T_{\rho_{L2}} = T_{\phi_{L1}} = T_{\phi_{L2}} = T$$



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Signal propagation errors: troposphere

- Refractivity $N = (n-1)\times 10^6$ $\quad \Delta\rho = \int_S^R [n(l)-1]\,dl = 10^{-6}\int_S^R N(l)\,dl$

$$N = N_d + N_w$$
$$T = 10^{-6}\int N(l)\,dl = 10^{-6}\int [N_d(l) + N_w(l)]\,dl = T_d + T_w$$

$$N_d \approx 77.64\frac{P}{T}$$
$$N_w \approx 3.73\cdot 10^5 \frac{e}{T^2}$$

P : total pressure [mbar]
T : temperature [K]
e : partial pressure water vapor [mbar]

if known → refractivity known

Signal propagation errors: troposphere

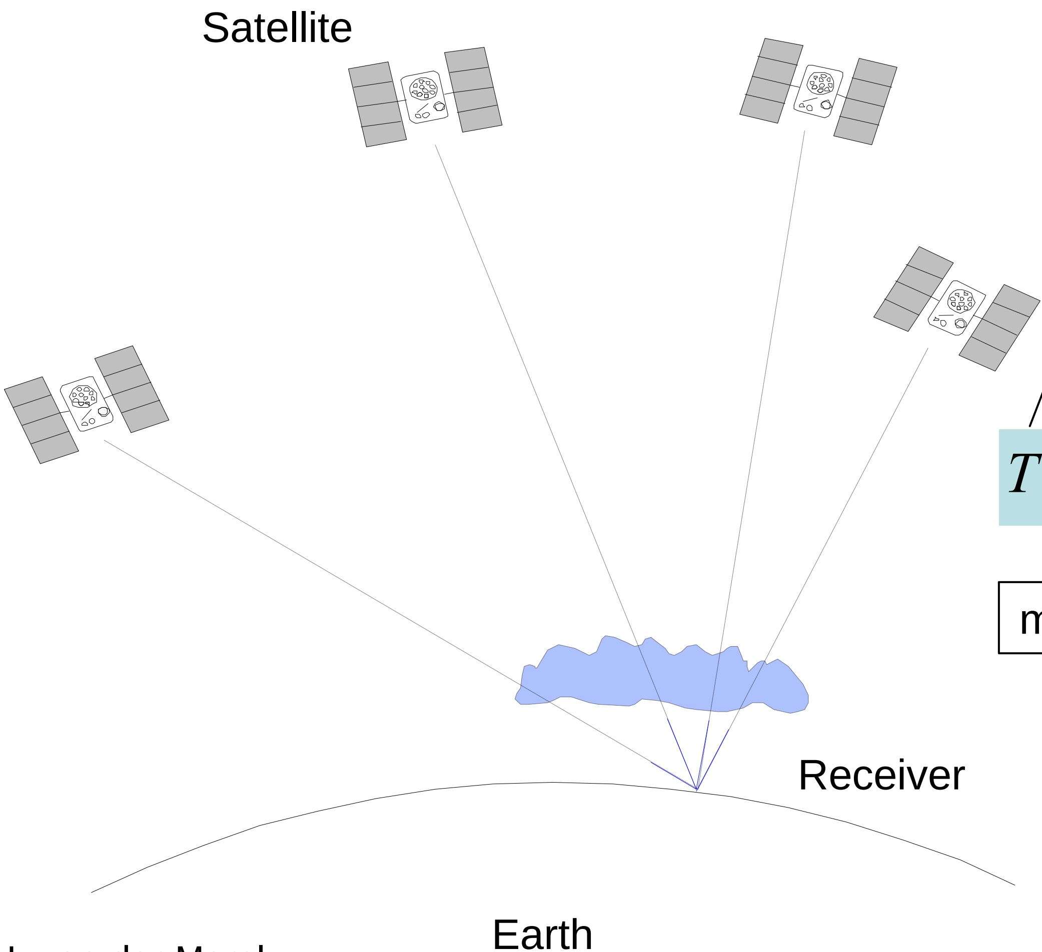


Figure: H. van der Marel

$$T = m_d(el) * T_{z,d} + m_w(el) * T_{z,w}$$

- tropospheric delay (T)
- computed hydrostatic delay ($m_d(el) * T_{z,d}$)
- mapping functions ($m_d(el)$, $m_w(el)$)
- Unknown tropospheric zenith wet delay ($T_{z,w}$)

11

Signal propagation errors: troposphere

- **Saastamoinen model**:
  zenith dry and wet delays calculated from temperature, pressure and humidity (measurements or standard atmosphere), height and latitude
- **Hopfield model**:
  dry and wet refractivities calculated

- Dry delay in zenith direction 2.3 – 2.6 m at sea level
  → can be predicted with accuracy of few mm's
- Wet delay depends on water vapor profile along path, 0 – 80 cm
  → accuracy of models few cm's
- If no actual meteorological observations available (standard atmosphere applied): total zenith delay error 5 – 10 cm

Signal propagation errors: summary

	ionosphere	troposphere
height	50 – 1000 km	0 – 16 km
variability	diurnal, seasonal, solar cycle (11 yr), solar flares	low
zenith delay	meters – tens of meters	2.3 – 2.6 m (sea level)
obliquity factor el=30°	1.8	2
obliquity factor el=15°	2.5	4
obliquity factor el= 3°	3	10
modeling error (zenith)	1 - >10 m	5 – 10 cm (no met. data)
dispersive	yes	no

all values are approximate, depending on location and circumstances

Signal propagation errors

Homework exercise:

- make plots of the different mapping functions (page 173 Misra and Enge) as function of the elevation angle (ranging from 0 – 90^{o})
- compare them with each other AND with the obliquity factor of the ionosphere delay (slide 22)
- try to explain the differences
- more details: see assignment on blackboard

Multipath

- Signal reflected: arrives via two or more paths at the antenna
- Reflected signals have different path length and interfere with direct signal
- Systematic error (does not average out)
  - pseudorange error: up to tens of meters
  - carrier phase error: up to 5 cm

Multipath

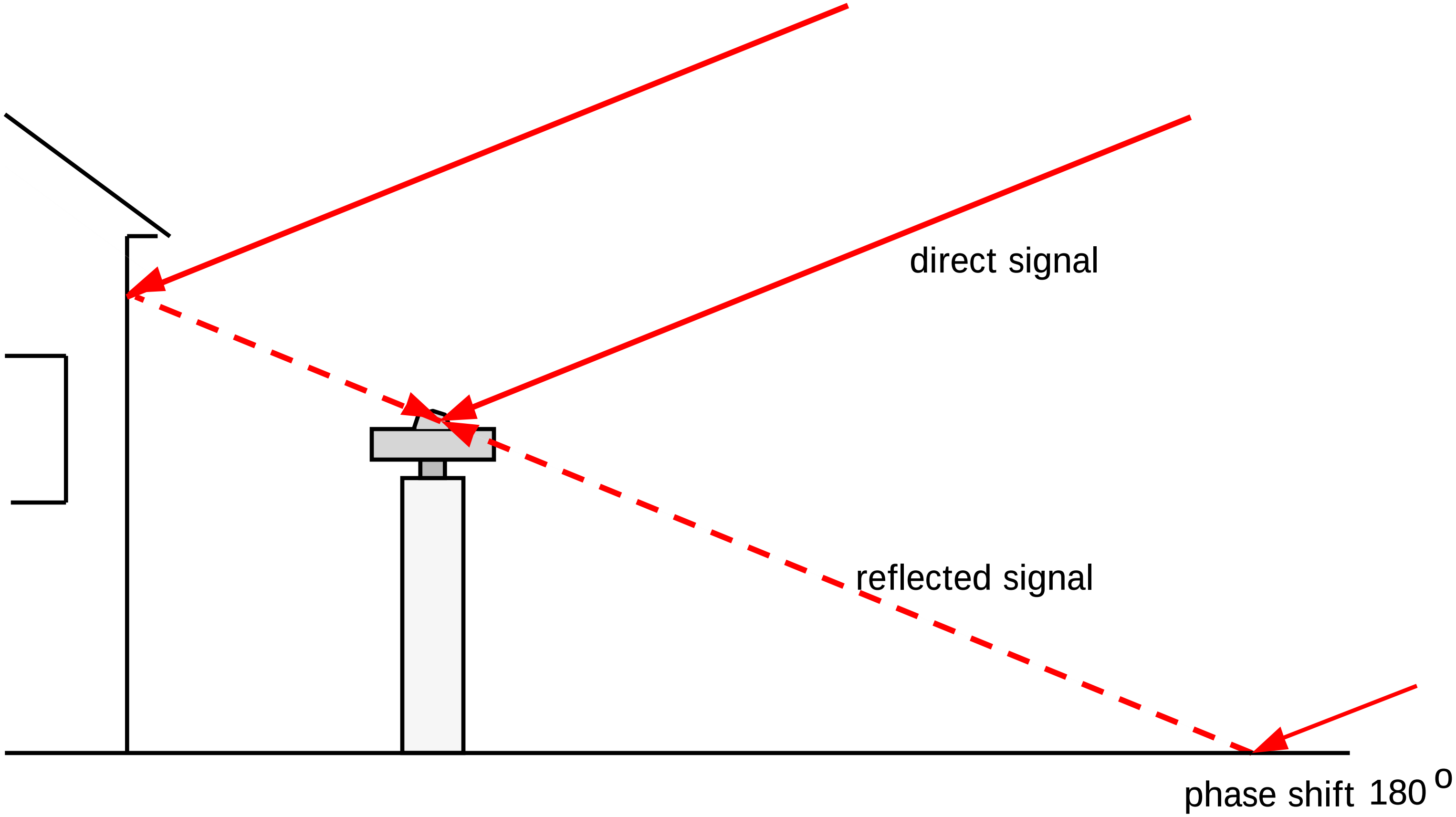


Figure: H. van der Marel

Multipath

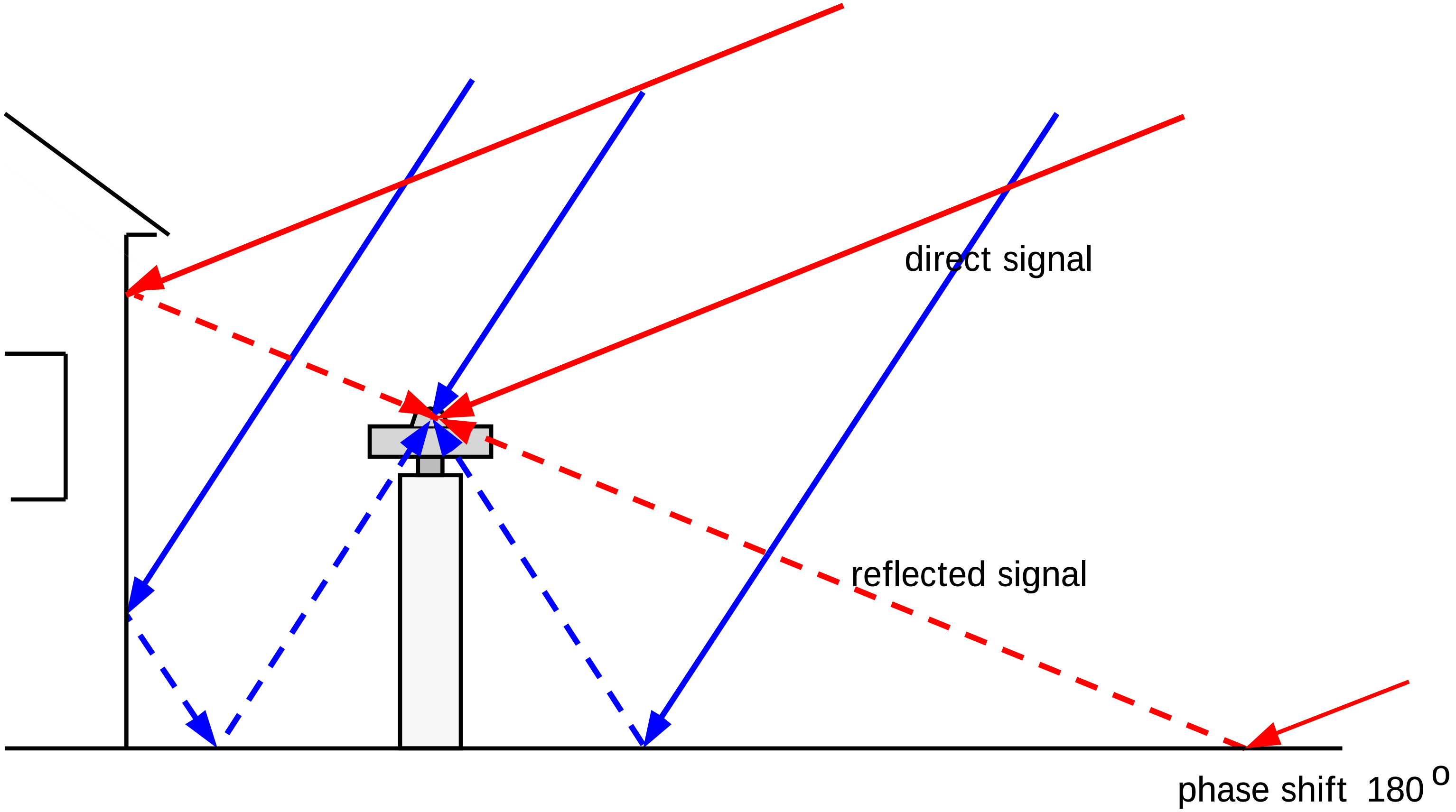


Figure: H. van der Marel

Multipath

Primary defense:

- careful selection of antenna locations (away from reflectors) → not always possible
- carefully designed antennas (choke rings; microstrip) → no signals from below
- signal processing (correlators)

Pseudorange multipath can be detected/analyzed by forming a special linear combination of code and carrier phase data

For GPS: if receiver is static, same multipath pattern repeats after 23h56m (same orbit)

Multipath: example

$$M_C = C1 - 4.092 \cdot L1 + 3.092 \cdot L2 \quad [m]$$

multipath on C/A-code pseudorange PRN28 between –7 and +6 m!

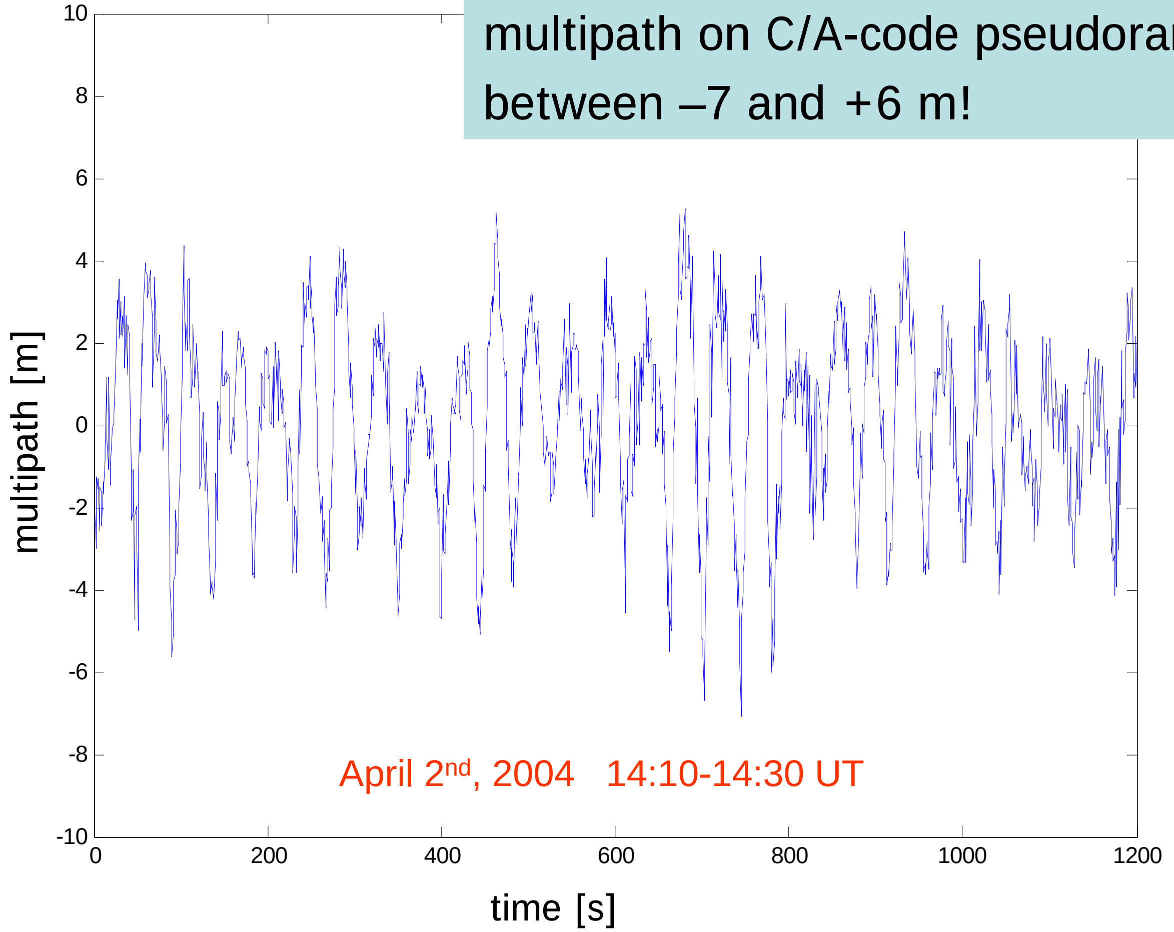


Figure: H. van der Marel

GPS error budget

Empirical values, actual values depend on receiver, atmosphere models, time and location

Error source	RMS range error [m]
satellite clock and ephemeris	$\sigma_{RE/CS} = 3$ m = SIS URE
atmospheric propagation modeling	$\sigma_{RE/P} = 5$ m
receiver noise and multipath	$\sigma_{RE/RNM} = 1$ m
User Range Error (URE)	$\sigma_{URE} = 6$ m

$$\sigma_{URE} = \sqrt{\sigma_{RE/CS}^2 + \sigma_{RE/P}^2 + \sigma_{RE/RNM}^2}$$

Carrier-smoothing

$$\Phi(t_i) = \underbrace{r(t_i) + c\cdot\left(\delta t_u(t_i) - \delta t^s(t_i - \tau_i)\right) + T(t_i)}_{=\rho_{IF}(t_i)} - I(t_i) + \lambda\cdot A + \varepsilon_\Phi(t_i)$$

precise estimate for change in pseudorange:

$$\Delta\Phi(t_i) = \Phi(t_i) - \Phi(t_{i-1}) = \Delta\rho_{IF}(t_i) - \Delta I(t_i) + \Delta\varepsilon_\Phi(t_i)$$

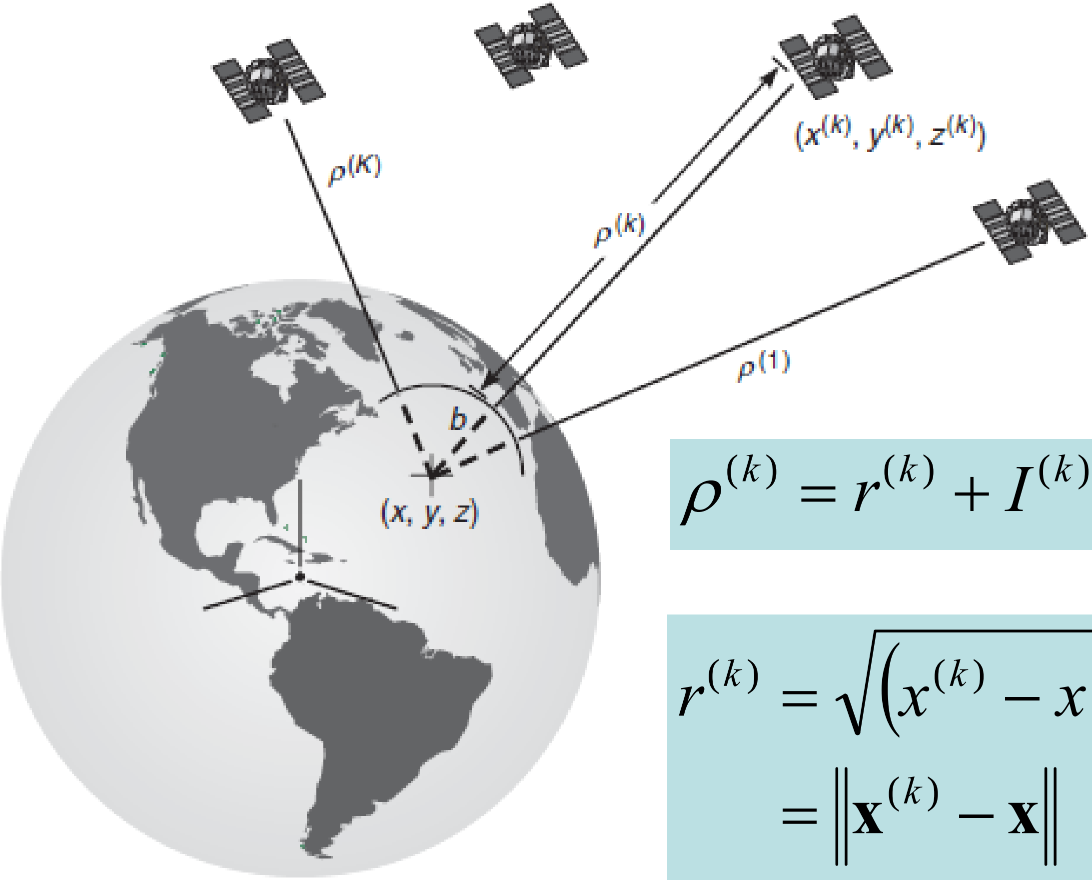
near zero if epochs close together

carrier-smoothed pseudorange
= weighted average of pseudorange (code) and carrier-derived pseudorange

$$\bar{\rho}(t_i) = \frac{1}{M}\rho(t_i) + \frac{M-1}{M}\left[\bar{\rho}(t_{i-1}) + \Delta\Phi(t_i)\right]$$

$$\bar{\rho}(t_1) = \rho(t_1)$$

Non-linear observation equations



$$\rho^{(k)} = r^{(k)} + I^{(k)} + T^{(k)} + c\left[\delta t_u - \delta t^{(k)}\right] + \varepsilon_\rho^{(k)}$$

$$r^{(k)} = \sqrt{\left(x^{(k)} - x\right)^2 + \left(y^{(k)} - y\right)^2 + \left(y^{(k)} - y\right)^2}$$
$$= \left\| \mathbf{x}^{(k)} - \mathbf{x} \right\|$$

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Non-linear observation equations

Corrected pseudorange: account for satellite clock offset, and compensate remaining error sources

Note: increased noise, since corrections/models are not perfect

$$\underbrace{}_{b}$$

$$\rho^{(k)} = r^{(k)} + c\delta t_u + \tilde{\varepsilon}_\rho^{(k)} = \left\| \mathbf{x}^{(k)} - \mathbf{x} \right\| + b + \tilde{\varepsilon}_\rho^{(k)}$$

$$\begin{aligned} r^{(k)} &= \sqrt{\left(x^{(k)} - x\right)^2 + \left(y^{(k)} - y\right)^2 + \left(y^{(k)} - y\right)^2} \\ &= \left\| \mathbf{x}^{(k)} - \mathbf{x} \right\| \end{aligned}$$

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Linearization

non-linear model

$$\mathbf{y} = \mathbf{H}(\mathbf{v}) + \boldsymbol{\varepsilon}$$

Taylor series

$$\mathbf{y} = \mathbf{H}(\mathbf{v}_0) + \frac{\partial \mathbf{H}(\mathbf{v}_0)}{\partial \mathbf{v}}\left(\mathbf{v} - \mathbf{v}_0\right) + \ldots + \boldsymbol{\varepsilon}$$

observed-minus-computed observations

$$\delta\mathbf{y} = \mathbf{y} - \mathbf{H}(\mathbf{v}_0)$$

correction to approximate values

$$\delta\mathbf{v} = \mathbf{v} - \mathbf{v}_0$$

design matrix

$$\mathbf{A} = \frac{\partial \mathbf{H}(\mathbf{v}_0)}{\partial \mathbf{v}}$$

linearized model:

$$\delta\mathbf{y} = \mathbf{A}\,\delta\mathbf{v} + \boldsymbol{\varepsilon}$$

Linearization

$$\mathbf{y} = \rho^{(k)} = \left\| \mathbf{x}^{(k)} - \mathbf{x} \right\| + b + \tilde{\varepsilon}_{\rho}^{(k)}$$

$$H(\mathbf{v}_0) = \rho_0^{(k)} = \left\| \mathbf{x}^{(k)} - \mathbf{x}_0 \right\| + b_0$$

$$\mathbf{y} = \mathbf{H}(\mathbf{v}_0) + \frac{\partial \mathbf{H}(\mathbf{v}_0)}{\partial \mathbf{v}} (\mathbf{v} - \mathbf{v}_0) + \ldots + \boldsymbol{\varepsilon}$$

Approximations required for:

- satellite position at time of transmission t - τ (from ephemeris) problem: τ not precisely known
- receiver position at time of reception t
- receiver clock error

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Linearization

$$\mathbf{y} = \rho^{(k)} = \left\|\mathbf{x}^{(k)} - \mathbf{x}\right\| + b + \tilde{\varepsilon}_{\rho}^{(k)}$$

$$\mathbf{y} = \mathbf{H}(\mathbf{v}_0) + \frac{\partial \mathbf{H}(\mathbf{v}_0)}{\partial \mathbf{v}}\left(\mathbf{v} - \mathbf{v}_0\right) + \ldots + \boldsymbol{\varepsilon}$$

$$H(\mathbf{v}_0) = \rho_0^{(k)} = \left\|\mathbf{x}^{(k)} - \mathbf{x}_0\right\| + b_0$$

$$\frac{\partial \mathbf{H}(\mathbf{v}_0)}{\partial \mathbf{v}} = \begin{bmatrix} \frac{\partial \mathbf{H}(\mathbf{v}_0)}{\partial x} \\ \frac{\partial \mathbf{H}(\mathbf{v}_0)}{\partial y} \\ \frac{\partial \mathbf{H}(\mathbf{v}_0)}{\partial z} \\ \frac{\partial \mathbf{H}(\mathbf{v}_0)}{\partial b} \end{bmatrix}^T = \begin{bmatrix} -\frac{x^{(k)} - x_0}{\left\|\mathbf{x}^{(k)} - \mathbf{x}_0\right\|} \\ -\frac{y^{(k)} - y_0}{\left\|\mathbf{x}^{(k)} - \mathbf{x}_0\right\|} \\ -\frac{z^{(k)} - z_0}{\left\|\mathbf{x}^{(k)} - \mathbf{x}_0\right\|} \\ 1 \end{bmatrix}^T = \begin{bmatrix} (-\mathbf{1}^{(k)})^T & 1 \end{bmatrix}$$

Linearized code observation equations

$$\boldsymbol{\delta\rho} = \begin{bmatrix} \delta\rho^{(1)} \\ \delta\rho^{(2)} \\ \vdots \\ \delta\rho^{(K)} \end{bmatrix} = \underbrace{\begin{bmatrix} (-\mathbf{1}^{(1)})^T & 1 \\ (-\mathbf{1}^{(2)})^T & 1 \\ \vdots & \vdots \\ (-\mathbf{1}^{(K)})^T & 1 \end{bmatrix}}_{\boldsymbol{G}} \begin{bmatrix} \delta\mathbf{x} \\ \delta b \end{bmatrix} + \tilde{\boldsymbol{\varepsilon}}_{\rho}$$

Least-squares estimation

linear model

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \boldsymbol{\varepsilon}; \qquad \mathbf{Q}_{\mathbf{yy}}$$

variance matrix

$$\hat{\mathbf{x}} = \underbrace{\left(\mathbf{A}^T \mathbf{Q}_{\mathbf{yy}}^{-1} \mathbf{A}\right)^{-1}}_{\mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}}} \mathbf{A}^T \mathbf{Q}_{\mathbf{yy}}^{-1} \mathbf{y}$$

Least-squares estimation

linearized model $\boldsymbol{\delta\rho} = \mathbf{G}\delta\mathbf{v} + \tilde{\boldsymbol{\varepsilon}}; \qquad \mathbf{Q}_{\rho\rho}$

iteration required, Gauss-Newton method:

$\mathbf{v_0} = \ldots$

while $\left\|\delta\hat{\mathbf{v}}\right\|^2_{Q_{\hat{v}\hat{v}}} \geq \eta$

$$\delta\boldsymbol{\rho} = \boldsymbol{\rho} - \mathbf{H}(\mathbf{v_0})$$

$$\mathbf{G} = \frac{\partial\mathbf{H}(\mathbf{v_0})}{\partial\mathbf{v}}$$

$$Q_{\hat{v}\hat{v}} = \left(\mathbf{G}^T\mathbf{Q}_{\rho\rho}^{-1}\mathbf{G}\right)^{-1}$$

$$\delta\hat{\mathbf{v}} = Q_{\hat{v}\hat{v}} \cdot \mathbf{G}^T\mathbf{Q}_{\rho\rho}^{-1}\delta\boldsymbol{\rho}$$

$$\hat{\mathbf{v}} = \mathbf{v_0} + \delta\hat{\mathbf{v}}$$

$$\mathbf{v_0} = \hat{\mathbf{v}}$$

end

Summary and outlook

- GPS measurements and error sources
- Linearized observation equations → position estimation

Next:

Position, Velocity and Time (PVT) estimation