Beams (Stringers, Stiffeners, Panel Breakers)

- axial (longitudinal) loads
- bending loads
- stiffening elements

- Different cross-sectional shapes
  - “L” or angle
  - “C” or channel
  - “Z”
  - “T” or blade
  - “I”
  - “J”
  - “Hat”
Beams...cross-section properties

- each section or member can have different layup =>
  - different stiffness
  - different strength
- more efficient structure by tailoring
# Beams...Layup Guidelines (qualitative)

<table>
<thead>
<tr>
<th>Location</th>
<th>Layup(^{(1)})</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flange away from skin</td>
<td>0 degree plies</td>
<td>need stiff material away from neutral axis</td>
</tr>
<tr>
<td>Web</td>
<td>±45 degree plies</td>
<td>buckling resistance under shear; high shear strength</td>
</tr>
<tr>
<td>Flange next to skin</td>
<td>0 degree plies and as close to the skin layup as possible</td>
<td>Stiff material away from neutral axis; reduced stiffness mismatch with skin</td>
</tr>
</tbody>
</table>

(1) 0 degrees aligned with axis of beam
Beams…Layup Guidelines (qualitative)

Load continuity issues

±45°

stiffness mismatch can lead to stiffener separation

$0_n$

A/0_p/A

Skin layup A/A

Bad Design!
First order correction of beam layup

- too many 0's implies microcracks
- impact damage resistance?
- is load transfer sufficient?
- is stiffness mismatch sufficiently reduced?

Improved design

approximation to skin layup

+45
-45
0
Second order correction of beam layup

• to be continued…
Beam cross section properties

- Equivalent axial stiffness \((EA)_{eq}\)
- From axial strain compatibility:

\[
\varepsilon_x^{(1)} = \varepsilon_x^{(2)} = \varepsilon_x^{(3)} = \varepsilon_a
\]

\[
\varepsilon_x^{(i)} = \frac{F_i}{(EA)_i} \quad \text{i=1-3}
\]

\[
(EA)_i = E_i b_i t_i
\]

\[
E_i = \frac{1}{(a_{11})_i t_i}
\]

- \(F_i\) is applied force on ith member
- \(E_i\) is \textbf{membrane} stiffness of ith member

note difference from isotropic case where \(E\) is not present!
Beam cross-section properties (axial loading)

- **Force sum:**
  \[ F_{TOT} = F_1 + F_2 + F_3 \]

- **Three eqns in the three unknowns** \( F_1 - F_3 \)
  \[
  F_i = \frac{(EA)_i}{\sum_{j=1}^{3}(EA)_j} F_{TOT} = \frac{E_i b_i t_i}{\sum_{j=1}^{3} E_j b_j t_j} F_{TOT}
  \]

- **Equivalent axial stiffness:**
  \[
  \varepsilon_a = \frac{F_{TOT}}{(EA)_{eq}} \Rightarrow (EA)_{eq} = \frac{F_{TOT}}{\varepsilon_a} \\
  \varepsilon_a = \frac{F_1}{(EA)_1} = \frac{(EA)_1}{(EA)_1 \sum (EA)_j} F_{TOT}
  \]
  \[
  (EA)_{eq} = \sum_j (EA)_j
  \]
Beam cross-section properties (bending load)

- Equivalent bending stiffness \((EI)_{eq}\)

\[
E_{b} \text{ not present in isotropic case!}
\]

Bending of all members is characterized by same radius of curvature (that of beam neutral axis):

\[
R_{ci} = R_{c1} = R_{c2} = R_{c3} = R_{ca}
\]

\[
R_{ci} = \frac{(EI)_{i}}{M_{i}}
\]

\[
(EI)_{i} = E_{bi} \left[ \frac{\text{(width)}_{i}\text{(height)}_{i}^{3}}{12} + A_{i}d_{i}^{2} \right]
\]

\[
E_{bi} = \frac{12}{t_{i}^{3}(d_{11})_{i}}
\]

\(E_{bi}\) is bending stiffness of \(i\)th member

BUT not for beams deeper than 1cm!
Beam Cross-section properties (bending load)

• **Moment sum:**

\[ M_{\text{TOT}} = M_1 + M_2 + M_3. \]

• **Three eqns in the three unknowns** \( M_1-M_3 \)

\[ M_i = \frac{(EI)_i}{\sum_{j=1}^{3} (EI)_j} M_{\text{TOT}} \]

analogous to the relation for the force on each member

• **Equivalent bending stiffness for cross-section:**

\[
\frac{M_1}{(EI)_1} = \frac{M_{\text{TOT}}}{(EI)_{eq}} \Rightarrow (EI)_{eq} = \frac{M_{\text{TOT}}}{M_1} (EI)_1 \\
M_1 = \frac{(EI)_1}{\sum_j (EI)_j} M_{\text{TOT}}
\]

\[(EI)_{eq} = \sum_j (EI)_j\]
Beam cross-sectional properties:

Example

- same as before but with bottom flange better defined

\[
\begin{array}{c|c|c|c|c|c}
\text{Member} & b (\text{mm}) & t (\text{mm}) & E_m (\text{GPa}) & E_b (\text{GPa}) \\
\hline
1 & 12.7 & 1.2192 & 75.6 & 32.4 \\
2 & 31.75 & 1.2192 & 18.2 & 17.9 \\
3 & 38.1 & 1.8288 & 56.5 & 47.9 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\text{Alum} & \text{Comp} & \Delta(\%) \\
\hline
E_A (\text{kN}) & 8525 & 5803 & 46.9 \\
E_I (\text{Nm}^2) & 1401 & 631 & 121.8 \\
\end{array}
\]

same thickness aluminum is more efficient but also 72% heavier calculated using \( E_b \)
Beam cross-sectional properties: Example (cont’d)

• increase (ply) thickness of composite by 46.9% and re-shuffle one flange layup:

<table>
<thead>
<tr>
<th>WAS</th>
<th>Layup</th>
<th>Member</th>
<th>b (mm)</th>
<th>t (mm)</th>
<th>Em (GPa)</th>
<th>Eb (GPa)</th>
<th>WAS</th>
</tr>
</thead>
<tbody>
<tr>
<td>[45/-45/02]s</td>
<td>[45/02/-45]s</td>
<td>1</td>
<td>12.7</td>
<td>1.791005</td>
<td>75.6</td>
<td>62.7</td>
<td>32.4</td>
</tr>
<tr>
<td>same</td>
<td>[45/-45/45/-45]s</td>
<td>2</td>
<td>31.75</td>
<td>1.791005</td>
<td>18.2</td>
<td>17.9</td>
<td>17.9</td>
</tr>
<tr>
<td>same</td>
<td>[45/-45/02/45/-45]s</td>
<td>3</td>
<td>38.1</td>
<td>2.686507</td>
<td>56.5</td>
<td>47.9</td>
<td>47.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Alum</th>
<th>Comp</th>
<th>Δ(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>EA (kN)</td>
<td>8525</td>
<td>8525</td>
<td>0.0</td>
</tr>
<tr>
<td>EI (Nm^2)</td>
<td>1401</td>
<td>1441</td>
<td>-2.8</td>
</tr>
</tbody>
</table>

\[
\frac{W_{Gr/Ep}}{W_{Al}} = 0.58 \frac{1.469}{1} = 0.852
\]

density ratio for carbon/epoxy

composite matches Al properties and is 15% lighter!
Beams - Column buckling

Both ends pinned

Both ends fixed

\[ P_{cr} = \frac{\pi^2 EI}{L^2} \]

\[ P_{cr} = \frac{4\pi^2 EI}{L^2} \]

EI calculated as before but with \( E_{\text{membrane}} \) for buckling calculation
**Column buckling – Effect of BCs and loading**

\[ P_{cr} = \frac{c \pi^2 EI}{L^2} \]

<table>
<thead>
<tr>
<th>Configuration</th>
<th>BC at left,right end</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>![pinned, pinned]</td>
<td>pinned, pinned</td>
<td>1.88</td>
</tr>
<tr>
<td>![fixed, fixed]</td>
<td>fixed, fixed</td>
<td>7.56</td>
</tr>
<tr>
<td>![fixed, pinned]</td>
<td>fixed, pinned</td>
<td>2.05</td>
</tr>
<tr>
<td>![fixed, pinned]</td>
<td>fixed, pinned</td>
<td>5.32</td>
</tr>
<tr>
<td>![fixed, free]</td>
<td>fixed, free</td>
<td>0.25</td>
</tr>
<tr>
<td>![fixed, free]</td>
<td>fixed, free</td>
<td>0.80</td>
</tr>
</tbody>
</table>

free: free rotation and free translation
pinned: free rotation, fixed translation
fixed: fixed rotation, fixed translation
Beam on elastic foundation under compression

- Foundation with spring constant $k$ (appropriate units)
- Ends with different translational and rotational spring constants (BC’s)

**Governing Equation**

$$EI \frac{d^4w}{dx^4} + P \frac{d^2w}{dx^2} + kw = 0$$
Beam on elastic foundation under compression

- can solve ODE which gives solutions of the form

\[ w = Ae^{px} \]

where

\[ p = \pm \sqrt{-\frac{P}{EI} \pm \sqrt{\left(\frac{P}{EI}\right)^2 - \frac{4k}{EI}}} \]

Combinations of sines, cosines, and exponentials depending on the magnitudes of \( P \) and \( k \)

satisfying the BC’s leads to a 4x4 eigenvalue problem in the buckling load \( P \)
Beam on elastic foundation under compression (pinned ends)

- or, which is faster, can use an energy approach

- The total energy in the beam is given by

\[ \Pi_c = \frac{1}{2} \int_0^L EI \left( \frac{d^2 w}{dx^2} \right)^2 dx + \frac{1}{2} \int_0^L (-P) \left( \frac{dw}{dx} \right)^2 dx + \frac{1}{2} \int_0^L kw^2 dx \]

- bending potential energy
- external work
- spring potential energy

- Assume deflection \( w \) in the form

\[ w = \sum A_m \sin \left( \frac{m\pi x}{L} \right) \]

Satisfies the BC’s

\( A_m \) are unknown coefficients

also happens to be the exact solution
Beam on elastic foundation under compression (pinned ends)

- Substitute in the energy expression and note the following:

\[
\left[ \sum A_m \sin \frac{m \pi x}{L} \right]^2 = \sum \sum A_m A_n \sin \frac{m \pi x}{L} \sin \frac{n \pi x}{L}
\]

\[
\sin \alpha \sin \beta = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \quad \alpha \neq \beta
\]

\[
= \frac{1}{2} (1 - \cos 2\alpha) \quad \alpha = \beta
\]

\[
\cos \alpha \cos \beta = \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)) \quad \alpha \neq \beta
\]

\[
= \frac{1}{2} (1 + \cos 2\alpha) \quad \alpha = \beta
\]
Beam on elastic foundation under compression (pinned ends)

- Energy expression becomes:

\[
\Pi_c = \sum \left[ \frac{(EI)m^4\pi^4}{4L^3} - \frac{Pm^2\pi^2}{4L} + \frac{kL}{4} \right] A_m^2
\]

- Minimizing with respect to \( A_m \):

\[
\frac{\partial \Pi_c}{\partial A_m} = 0
\]

- leads to:

\[
2 \left[ \frac{(EI)m^4\pi^4}{4L^3} - \frac{Pm^2\pi^2}{4L} + \frac{kL}{4} \right] A_m = 0
\]
Beam on elastic foundation under compression (pinned ends)

- which after rearranging reads

\[ \begin{bmatrix} \frac{\pi^2 EI}{L^2} \left( m^2 + \frac{kL^4}{\pi^4 (EI)m^2} \right) \end{bmatrix} - P \begin{bmatrix} A_m \end{bmatrix} = 0 \]

\( \text{diagonal matrix} \)

\( \text{column vector} \)

- either \( A_m = 0 \) (trivial solution, no bending)
- or the determinant of the diagonal matrix = 0

- set

\[ K_{mm} = \frac{\pi^2 EI}{L^2} \left( m^2 + \frac{kL^4}{\pi^4 (EI)m^2} \right) \]
Solution (buckling load of beam on elastic foundation)

\[
\begin{bmatrix}
K_{11} - P & 0 & 0 & 0 & 0 \\
0 & K_{22} - P & 0 & 0 & 0 \\
0 & 0 & K_{33} - P & 0 & 0 \\
0 & 0 & 0 & ... & ... \\
0 & 0 & 0 & ... & ... \\
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
... \\
... \\
... \\
\end{bmatrix} = 0
\]

\[
(K_{11} - P)(K_{22} - P)(K_{33} - P)... = 0
\]

\[
P_{cr} = \min(K_{ii})
\]
Special case: $k=0$

- pinned beam under compression

$$P_{cr} = K_{mn} = \frac{\pi^2 EI}{L^2} \left( m^2 + \frac{kL^4}{\pi^4 EI m^2} \right)$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} \frac{m^2}{m^2}$$

- minimized for $m=1 \Rightarrow$ exact solution for buckling load
Beam on elastic foundation under compression (pinned ends)

- the dependence of the buckling load on the spring constant $k$ can be seen more easily in a graph.

Rearranging:

$$\frac{P_{cr}}{\pi^2 EI} = m^2 + \frac{kL^4}{\pi^4 EI} \frac{1}{m^2}$$
Beam on elastic foundation: other BC’s


- general BC’s at the two ends (moment and shear force balance):

\[
\begin{align*}
- EI \frac{d^2 w}{dx^2} + G_1 \frac{dw}{dx} &= 0 \\
EI \frac{d^3 w}{dx^3} + P \frac{dw}{dx} + K_1 w &= 0
\end{align*}
\]  
\text{at } x=0

\[
\begin{align*}
- EI \frac{d^2 w}{dx^2} + G_2 \frac{dw}{dx} &= 0 \\
EI \frac{d^3 w}{dx^3} + P \frac{dw}{dx} + K_2 w &= 0
\end{align*}
\]  
\text{at } x=L
Beam on elastic foundation: Other BC’s

• Define:

\[ R_i = \frac{G_i L}{EI} \]

\[ \rho_i = \frac{1}{1 + \frac{3}{R_i}} \]

\[ \begin{cases} 
\rho=1 \Rightarrow \text{fixed} \\
\rho=0 \Rightarrow \text{pinned} 
\end{cases} \]

• \( w=0 \) at both ends; \( dw/dx \) specified at one end \( (0 \leq \rho_2 \leq 1) \) and pinned at the other \( (\rho_1=0) \)
Beam on elastic foundation: other BC’s

\[ \frac{P_{cr}}{\pi^2 EI L^2} = y \]

\[ \frac{\sqrt{kL^4}}{EI} = x \]

\[ \rho_1 = 0 \]

different values of \( \rho_2 \)

see next Figure
Beam on elastic foundation: other BC’s

\[ \frac{P_{cr}}{\pi^2 EI \frac{L^2}{k}} = y \]

\[ \sqrt{\frac{kL^4}{EI}} = x \]

\[ \rho_1 = 0 \]

\[ \rho_2 = 0 \]

\[ \rho_2 = 0.5 \]

\[ \rho_2 = 1 \]
### Beam on elastic foundation: Other BC’s

<table>
<thead>
<tr>
<th>$\rho_2$</th>
<th>$x$</th>
<th>$y$</th>
<th>$r^2$ (goodness of fit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0≤$x$≤20</td>
<td>$0.0099x^2 + 0.0041x + 1$</td>
<td>1.0000</td>
</tr>
<tr>
<td>0</td>
<td>20&lt;$x$≤100</td>
<td>$0.0004x^2 + 0.1517x + 1.6658$</td>
<td>0.9987</td>
</tr>
<tr>
<td>0.5</td>
<td>0≤$x$≤20</td>
<td>$0.0084x^2 + 0.0169x + 1.4069$</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.5</td>
<td>20&lt;$x$≤100</td>
<td>$0.0002x^2 + 0.1714x + 1.4518$</td>
<td>0.9988</td>
</tr>
<tr>
<td>1</td>
<td>0≤$x$≤20</td>
<td>$0.0069x^2 + 0.01134x + 2.046$</td>
<td>1.0000</td>
</tr>
<tr>
<td>1</td>
<td>20&lt;$x$≤100</td>
<td>$7x10^{-5}x^2 + 0.1924x + 1.2722$</td>
<td>0.9998</td>
</tr>
</tbody>
</table>
# Beam on elastic foundation: Other BC’s

<table>
<thead>
<tr>
<th>$\rho_1 = \rho_2$</th>
<th>$x$</th>
<th>$y$</th>
<th>$r^2$ (goodness of fit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>$0 \leq x \leq 20$</td>
<td>$0.0099x^2 + 0.0039x + 1.28$</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.2</td>
<td>$20 &lt; x \leq 100$</td>
<td>$0.0004x^2 + 0.1517x + 1.9512$</td>
<td>0.9987</td>
</tr>
<tr>
<td>0.5</td>
<td>$0 \leq x \leq 20$</td>
<td>$0.0099x^2 + 0.0019x + 1.916$</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.5</td>
<td>$20 &lt; x \leq 100$</td>
<td>$0.0003x^2 + 0.1539x + 2.5361$</td>
<td>0.999</td>
</tr>
<tr>
<td>1</td>
<td>$0 \leq x \leq 20$</td>
<td>$0.0051x^2 + 0.0265x + 4$</td>
<td>1.0000</td>
</tr>
<tr>
<td>1</td>
<td>$20 &lt; x \leq 100$</td>
<td>$-0.0003x^2 + 0.2385x + 2.3368$</td>
<td>0.9943</td>
</tr>
</tbody>
</table>
Beams under combined compression and bending

Governing equation (same as for beam on elastic foundation with $k=0$):

$$EI \frac{d^4w}{dx^4} + P \frac{d^2w}{dx^2} = 0$$

Solution

$$w = C_o + C_1 \sin \left( \sqrt{\frac{P}{EI}} x \right) + C_2 \cos \left( \sqrt{\frac{P}{EI}} x \right) + C_3 x$$

$C_o$, $C_1$, $C_2$, and $C_3$ determined from BC’s
Beams under combined compression and bending

• “Traditional” superposition is NOT applicable

• nor can one separate bending from axial deformations:

• why is the problem non-linear?
Beams under combined compression and bending

• special type of “superposition” is applicable:
  – since the source of non-linearity is in the axial load, if the axial load is kept equal to $P$ in each constituent problem, superposition is valid

Example:

The full compressive load applied on each case