AE4536: Buckling of structures

What is stability ?

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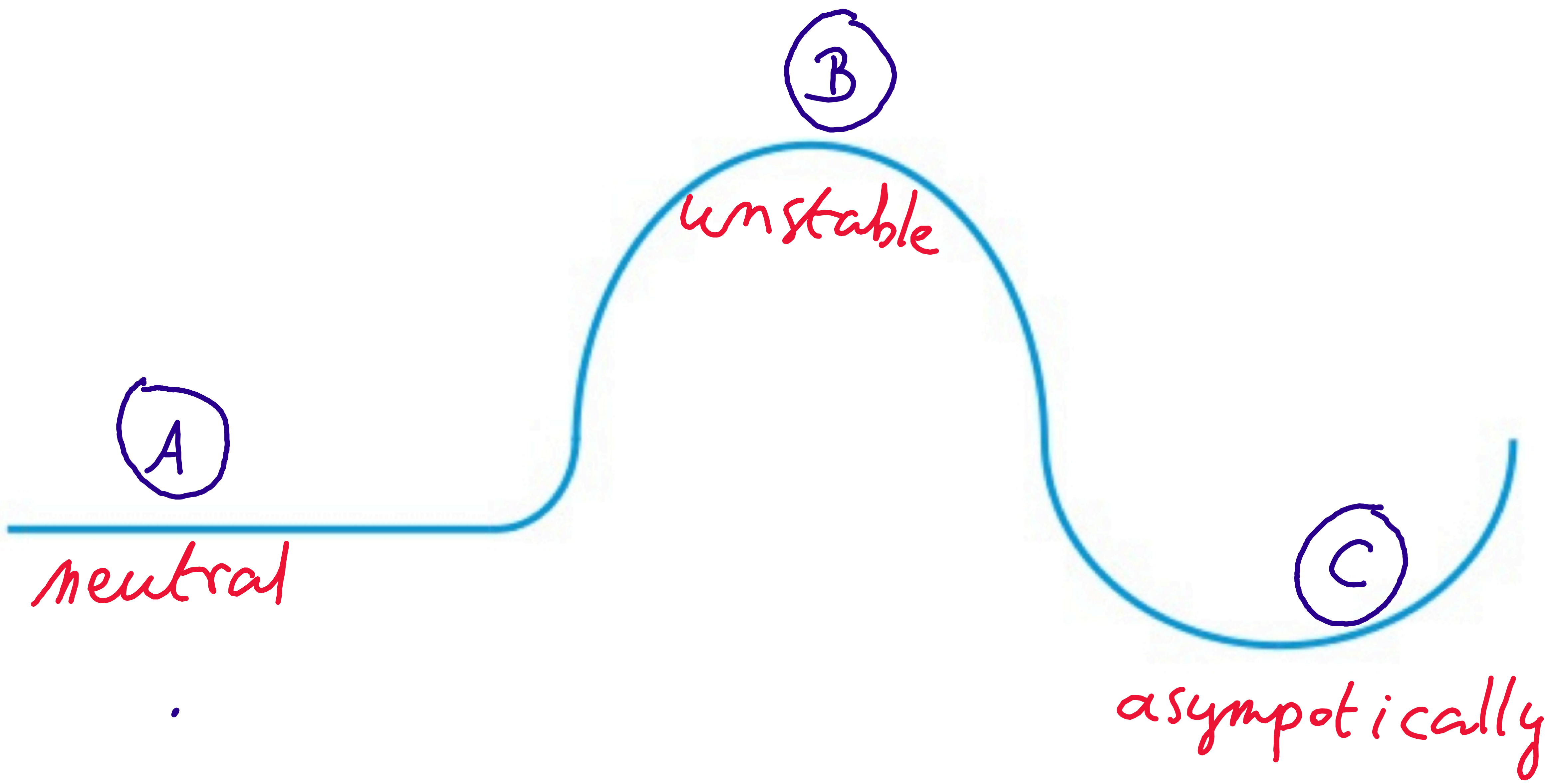
Learning goals

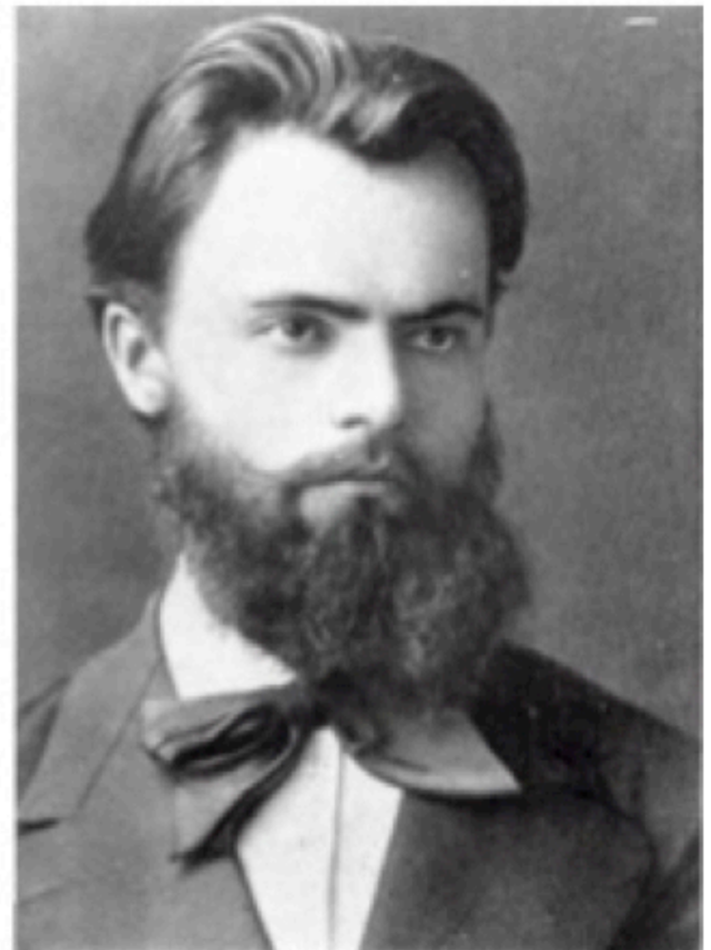
- Concept of stability
- Types of stability of linear systems
- Stability of nonlinear linearised systems
- Liapunov criteria for stability
- Importance of potential energy

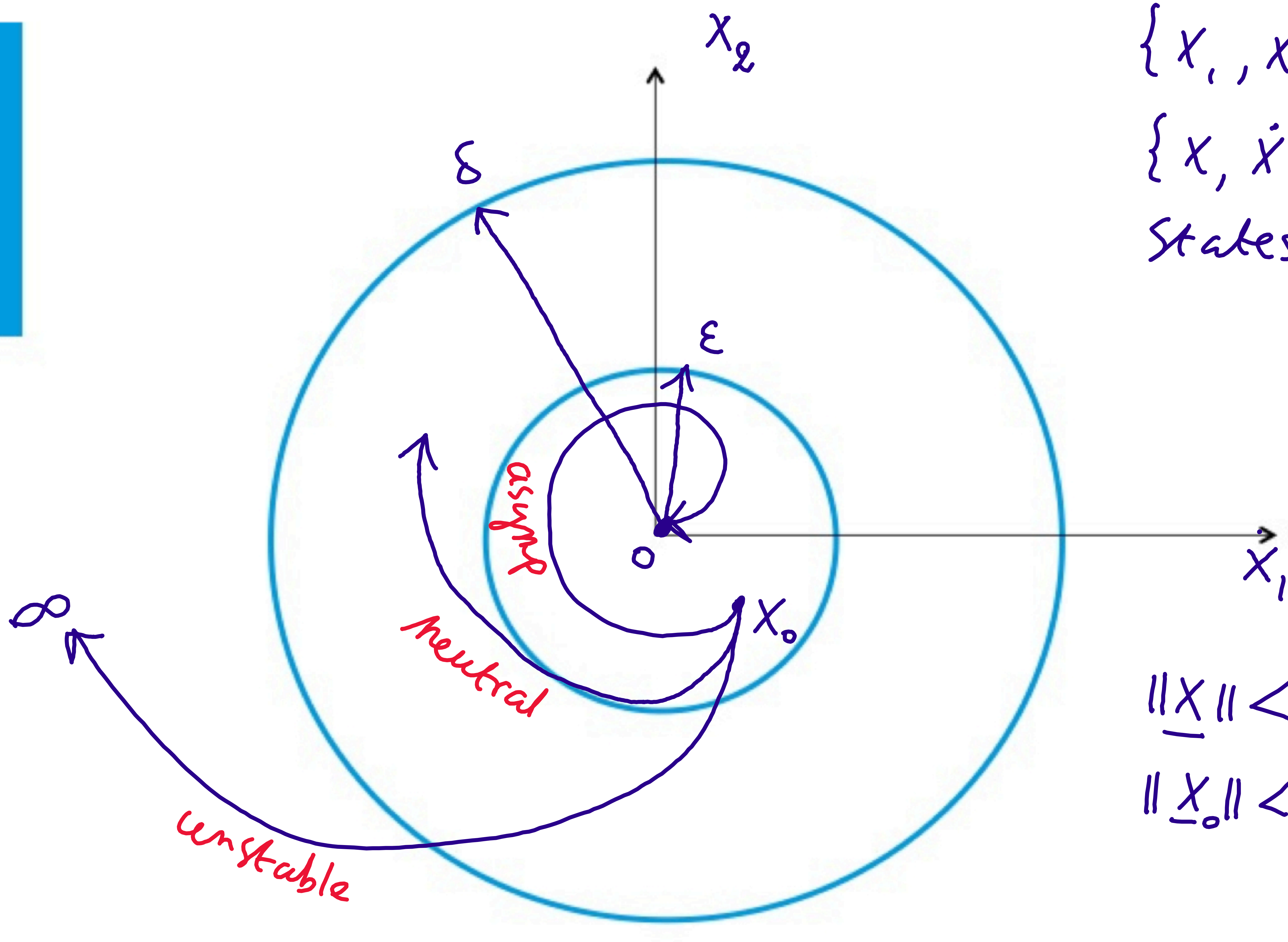
Ball analogy



Liapunov

- Alexandr Liapunov: 1857 – 1918
- Not to confuse with Sergei Liapunov, Russian composer and pianist
- Student of Chebyshev
- Defended a MSc thesis in 1884 *On the stability of ellipsoidal forms of equilibrium of rotating fluids*
- Defended a PhD thesis in 1892 on *The general problem of the stability of motion*





$\{x_1, x_2\}$

$\{x, \dot{x}\}$

states

$\|\underline{x}\| < \delta$

$\|\underline{x}_0\| < \varepsilon$

$\downarrow$

$\varepsilon(\delta)$

$$\begin{cases} \dot{X}_1 = a_{11} X_1 + a_{12} X_2 \\ \dot{X}_2 = a_{21} X_1 + a_{22} X_2 \end{cases}$$

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$\dot{\underline{X}} = A(\underline{X})$$

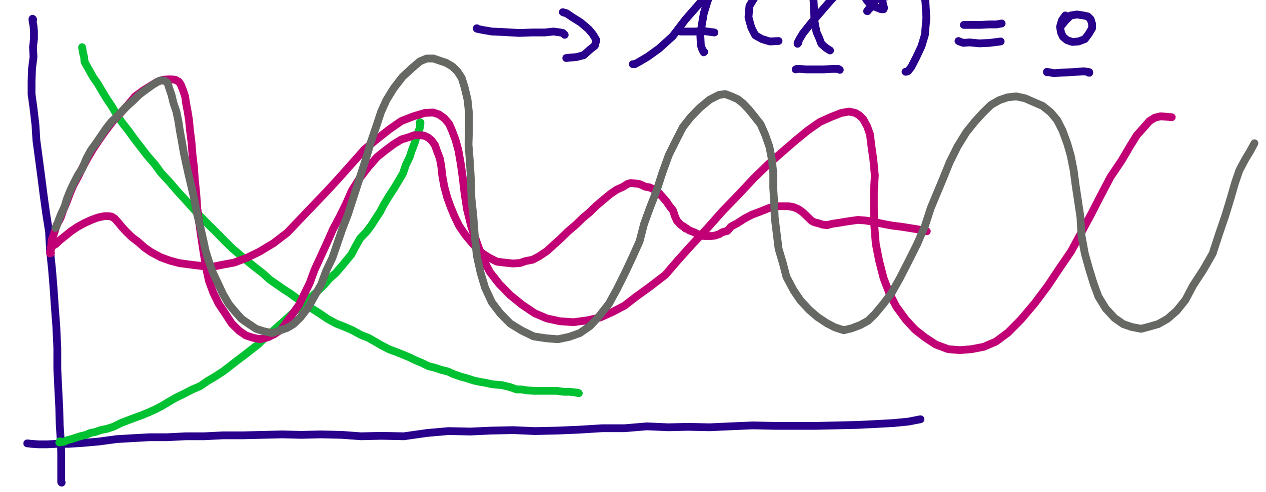
$$A(\underline{a}) = \underline{0} \rightarrow \underline{X}^* = \underline{X} - \underline{a}$$

$$\rightarrow A(\underline{X}^*) = \underline{0}$$

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} e^{\lambda t}$$

$$\lambda \rightarrow \sigma + i\omega$$

$\rightarrow$

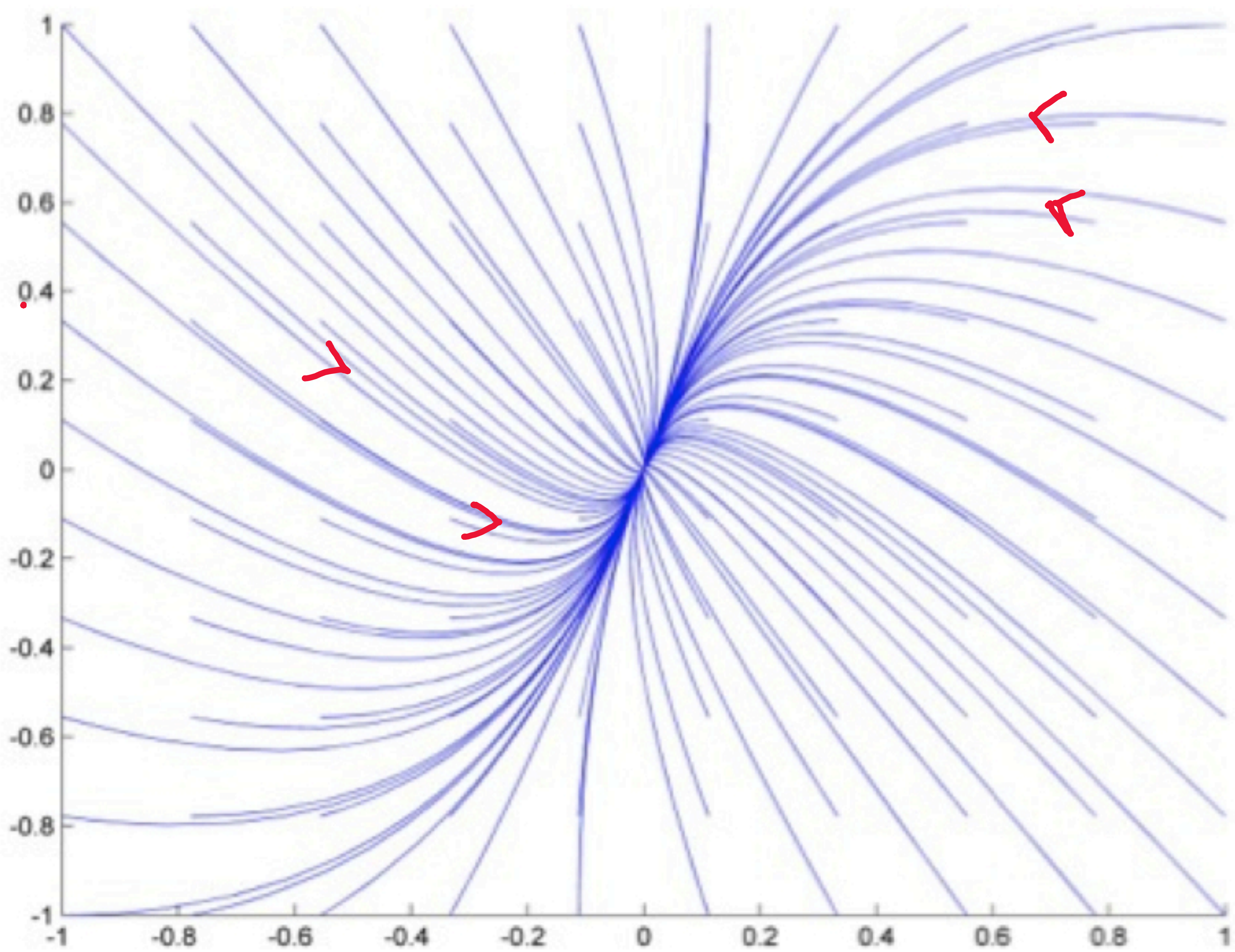


a_{11} . . .

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \lambda = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \rightarrow \underbrace{\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}}_{\det = 0} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \underline{0}$$

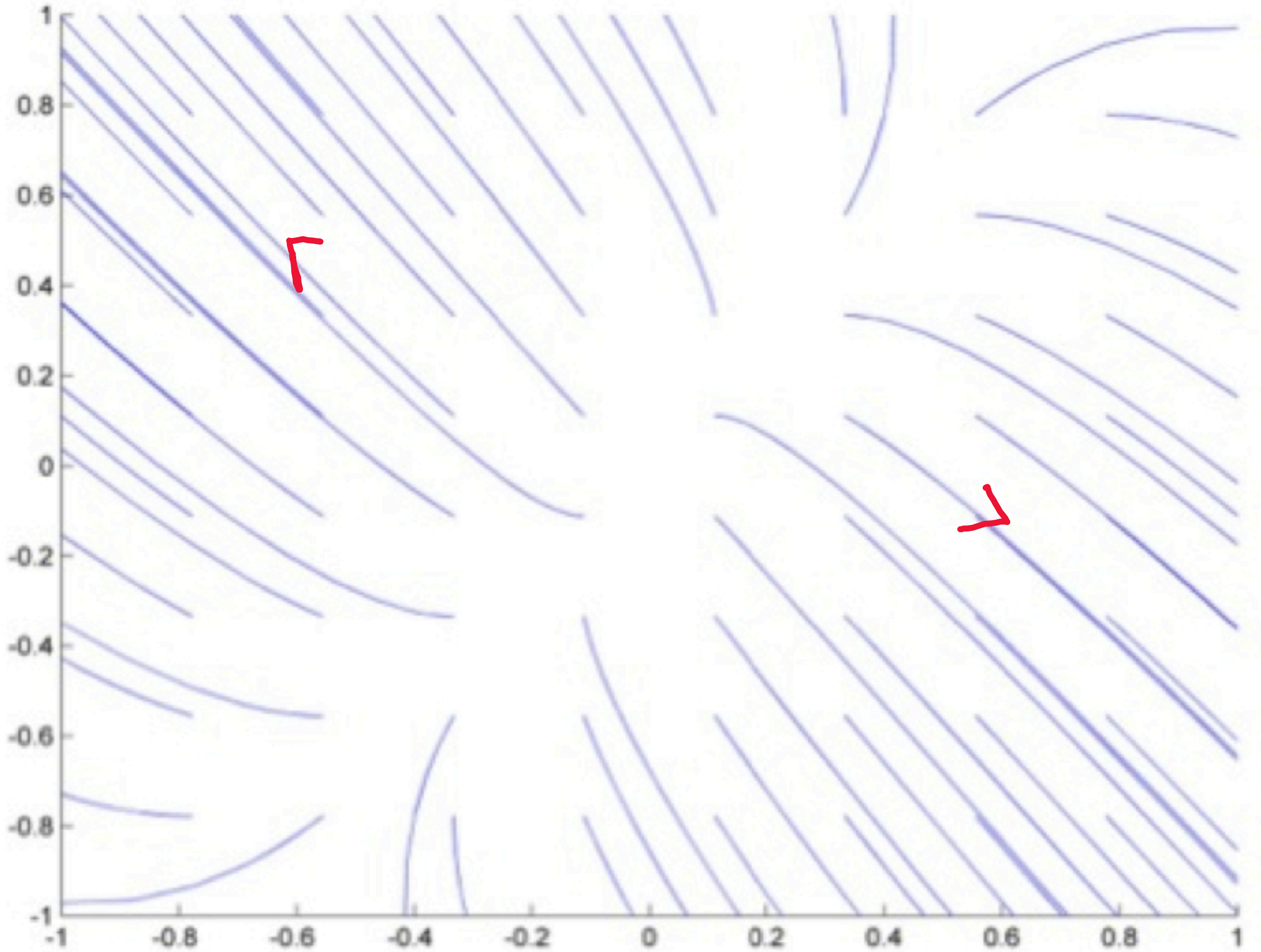
Stable node

λ real
$\lambda < 0$



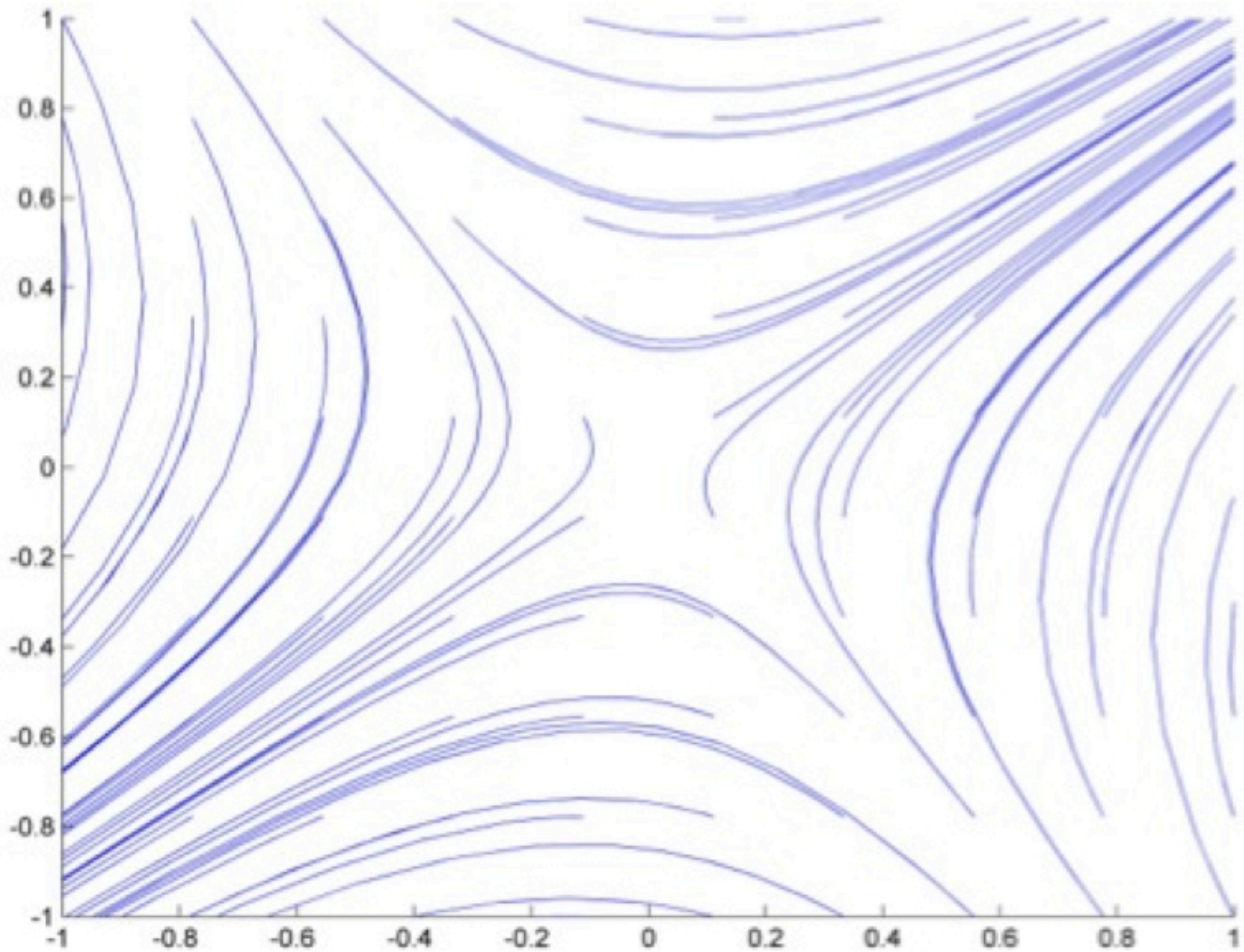
Unstable node

λ real
$\lambda > 0$



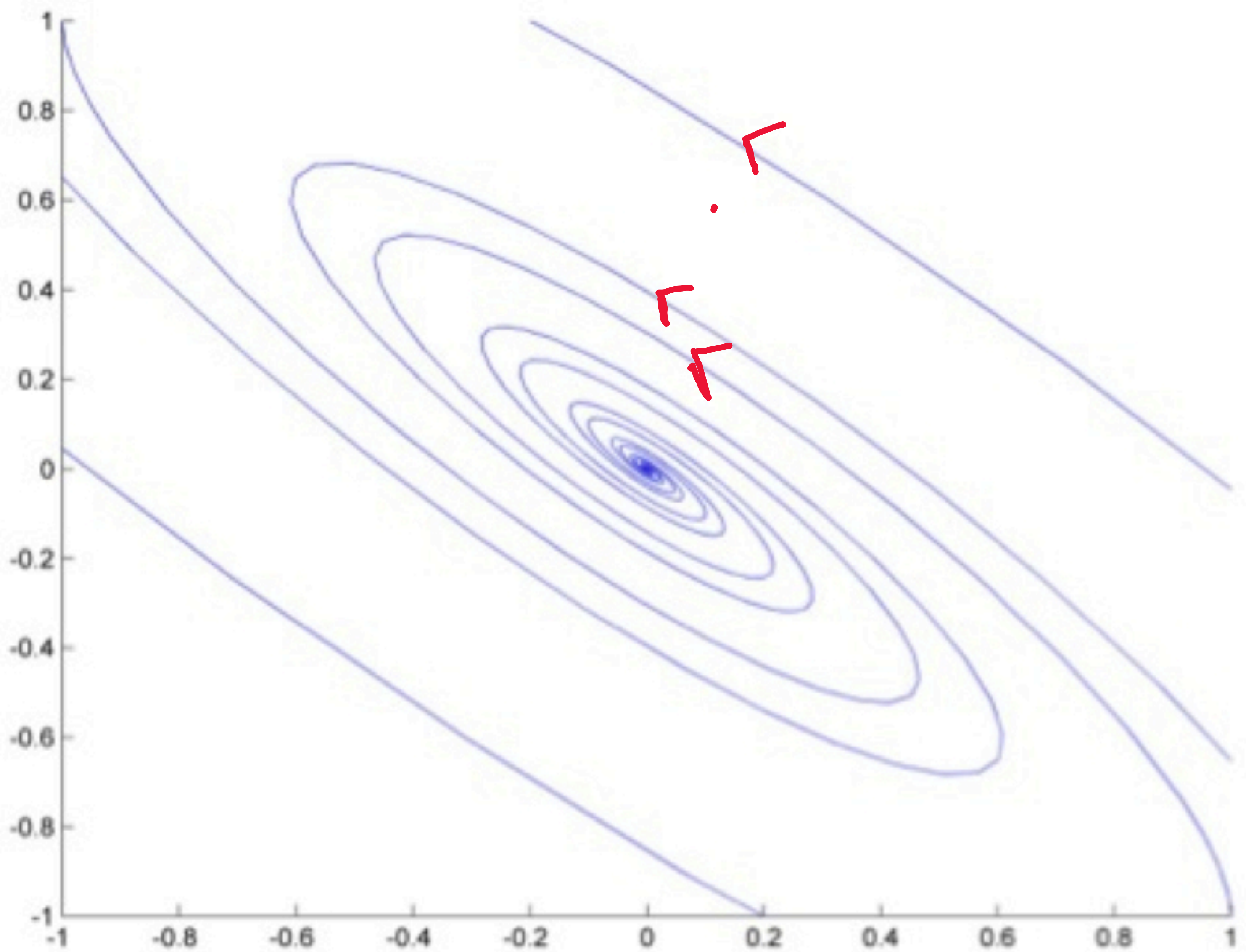
Saddle point

λ real
$\lambda_1 > 0$
$\lambda_2 < 0$



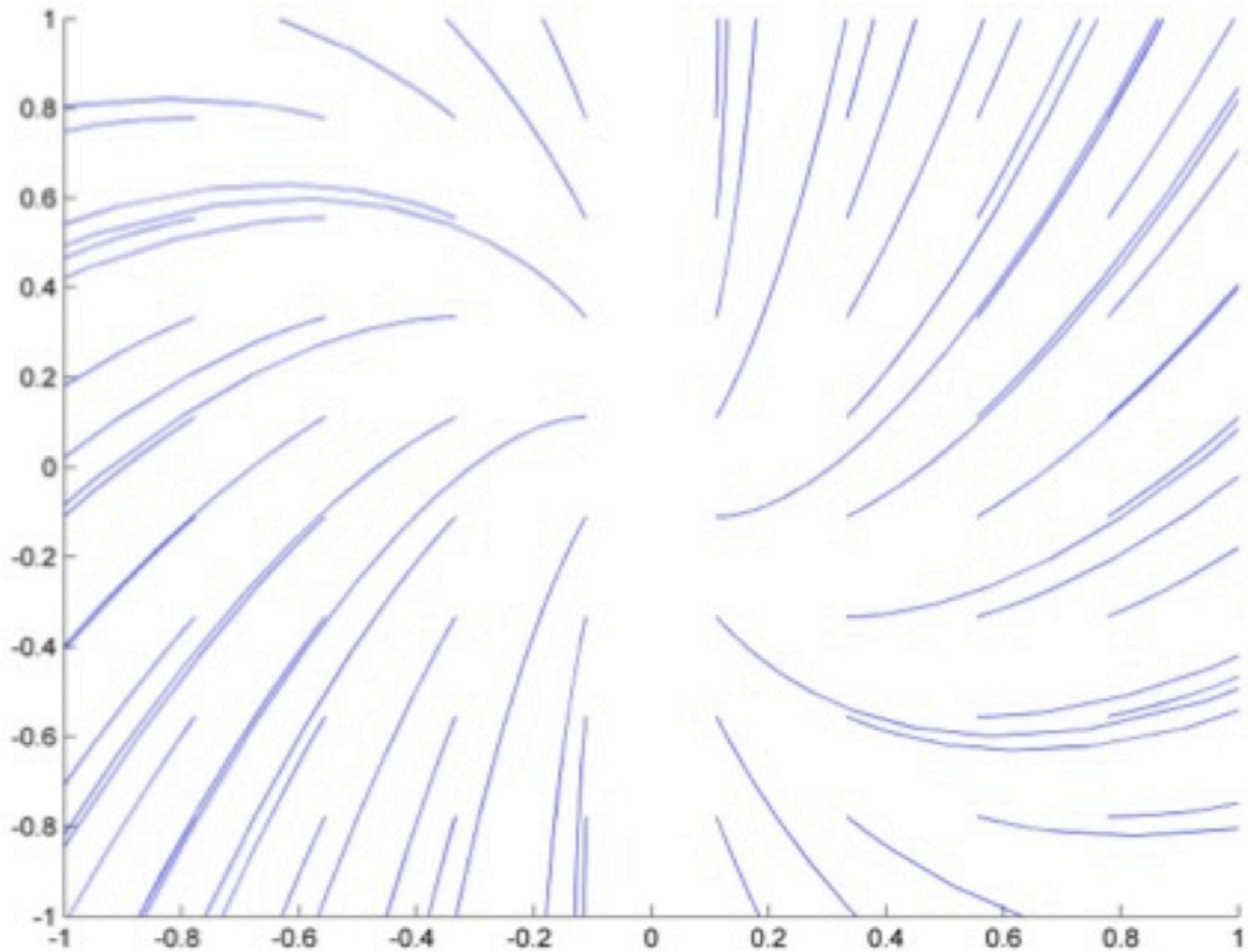
Stable focus

λ complex
$Re(\lambda) < 0$



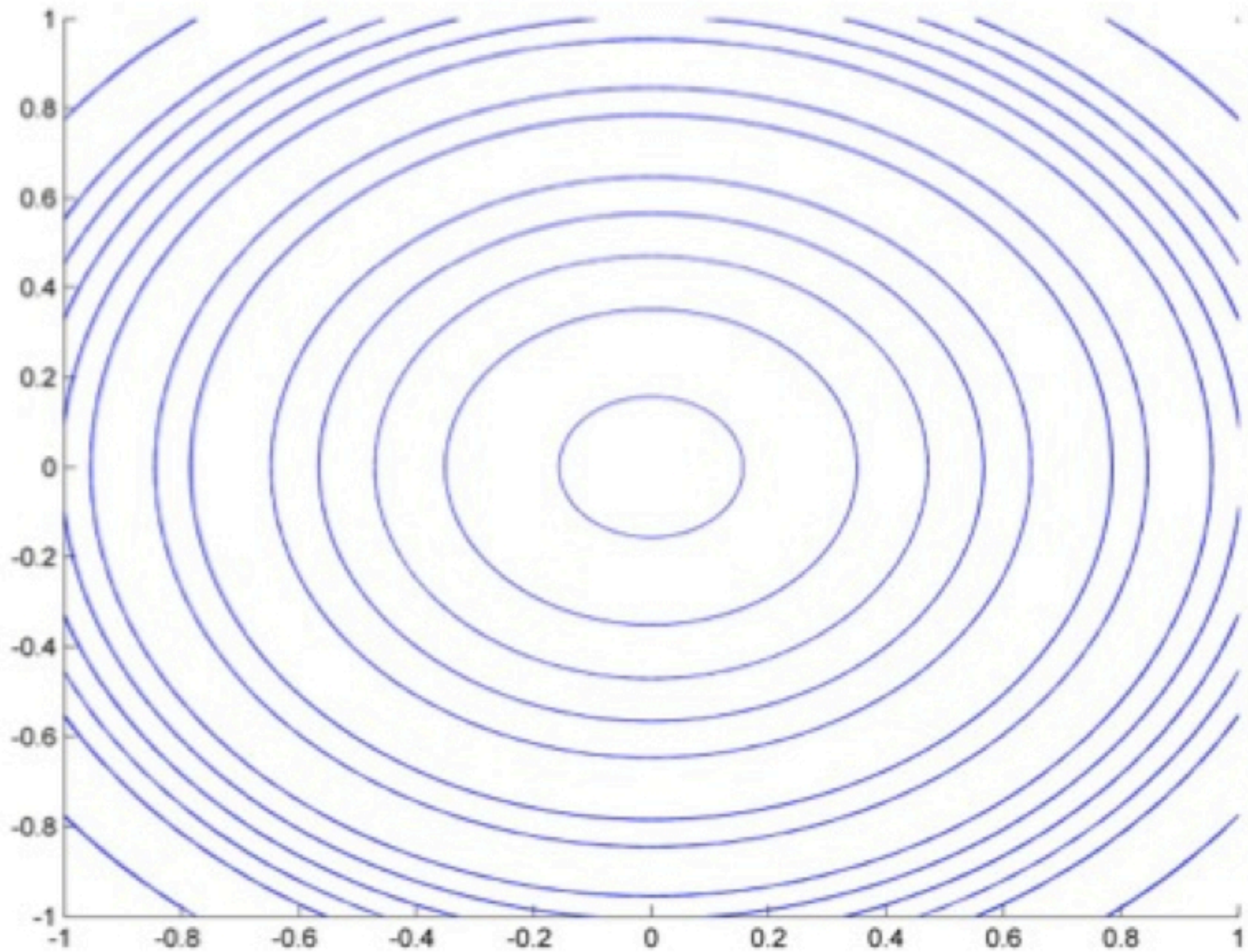
Unstable focus

λ complex
$Re(\lambda) > 0$



λ imaginary

Stable centre



Linear stability

$\lambda_1 < 0 \quad \wedge \quad \lambda_2 < 0$	Asymptotically stable
$\lambda_1 > 0 \quad \wedge \quad \lambda_2 > 0$	Unstable
$\lambda_1 < 0 \quad \wedge \quad \lambda_2 > 0$	Unstable
$\mathrm{Re}(\lambda_1) < 0$	Asymptotically stable
$\mathrm{Re}(\lambda_1) > 0$	Unstable
$\mathrm{Re}(\lambda_1) = 0$	Neutrally stable

$$\dot{\underline{x}} = \underset{\substack{\uparrow \\ \text{lin}}}{A(\underline{x})} + \underset{\substack{\uparrow \\ \text{non lin}}}{R(\underline{x})} \qquad R(\underline{x}_0) = O(\underline{x}_0)$$

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} -2x + 2\sin y \\ -2x - \cos y + e^{y} \end{Bmatrix} \qquad \begin{Bmatrix} x \\ y \end{Bmatrix}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{cases} \dot{x} = -2x + 2\left(y - \cancel{\frac{1}{3!} y^3}\right) + h.o.t. \\ \dot{y} = -2x - \left(\cancel{1} - \cancel{\frac{1}{2!} y^2}\right) + \cancel{1} + y + \cancel{\frac{1}{2!} y^2} + h.o.t \end{cases}$$

$$\begin{cases} \dot{x} = -2x + 2y \\ \dot{y} = -2x + y \end{cases}$$

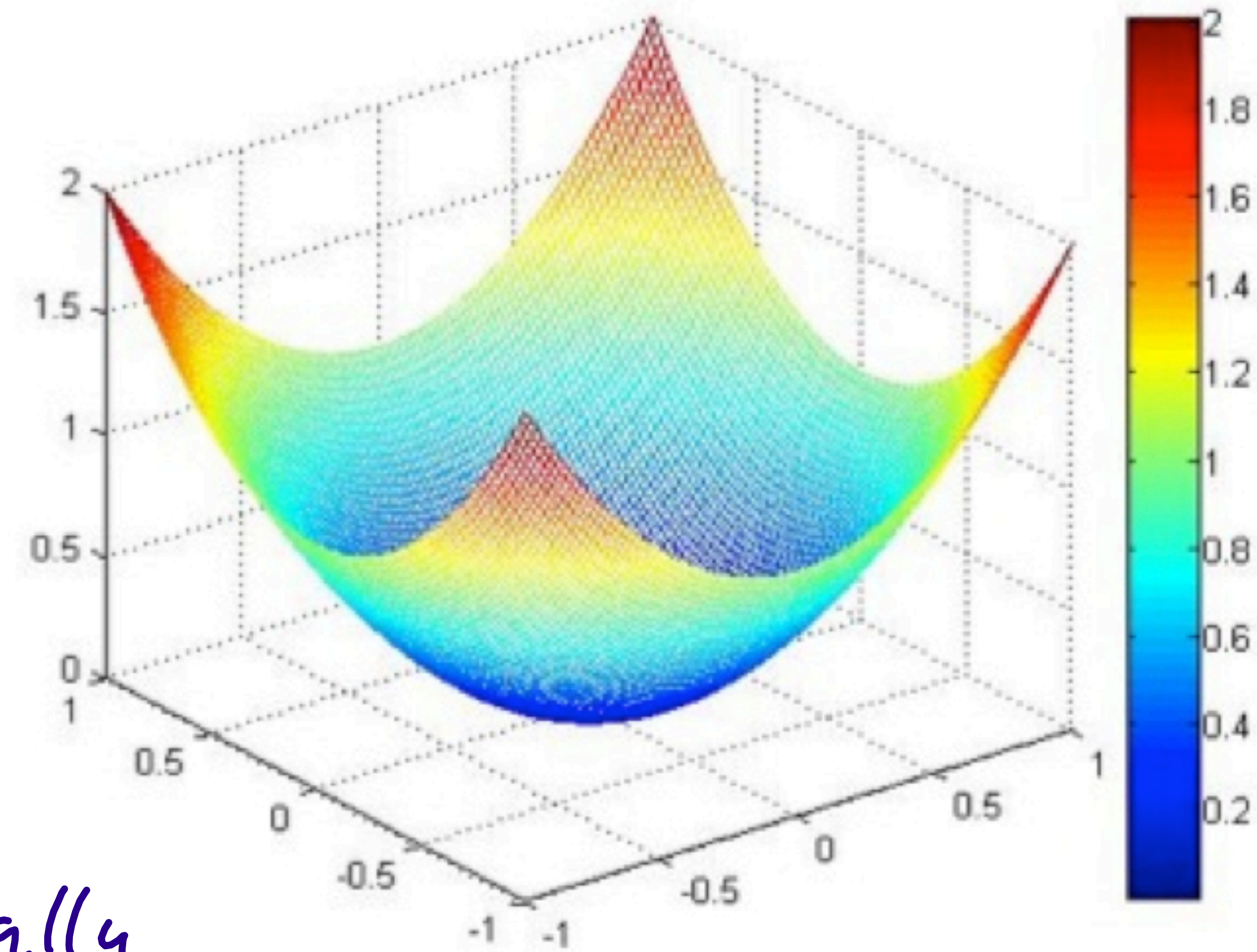
Asymptotically $\longrightarrow$ ok

Neutrally $\longrightarrow$ no judgement

The Liapunov function

P O S i T i V E

- Function V and its derivatives are continuous near the origin
- $V(0) = 0$
- Function V has an isolated minimum at the origin
- $\dot{V} \leq 0$



$\dot{V} = 0$ neutral

$\dot{V} < 0$ asymptotically

$$m\ddot{x} + kx = 0$$

$$\begin{cases} m\dot{x}_1 = x_2 \\ \dot{x}_2 = -kx_1 \end{cases}$$

$$V = a_1 x_1^2 + a_2 x_2^2$$

$$\dot{V} = 2a_1 x_1 \dot{x}_1 + 2a_2 x_2 \dot{x}_2$$

$$= 2a_1 x_1 \left(\frac{x_2}{m}\right) + 2a_2 x_2 (-kx_1) \leq 0$$

$$a_1 = \frac{1}{2}k \;,\; a_2 = \frac{1}{2}\frac{1}{m}$$

$$V = \underbrace{\frac{1}{2}kx_1^2}_{\text{Potential}} + \underbrace{\frac{1}{2}\frac{x_2^2}{m}}_{\text{Kinetic}} = E \qquad \dot{E} = 0$$

$$\begin{cases} m\dot{x}_1 = x_2 & (1) \\ \dot{x}_2 = -f(x_1) & (2) \end{cases}$$

$$m\ddot{x} + \underbrace{f(x)}_{\hookrightarrow \text{nonlinear}} = 0$$

$$(1):(2) \quad m\frac{dx_1}{dx_2} = -\frac{x_2}{f(x_1)}$$

$$\underbrace{\frac{1}{2}\frac{x_2^2}{m}}_{Kin} + \underbrace{\int f(x_1)\,dx_1}_{P(x_1)} = E$$

Is $P(x_1)$ positive definite ?

Summary

- Concept of stability
    - A system in equilibrium can be unstable, neutrally stable or asymptotically stable
    - The stability of a system is only valid in a certain region around an equilibrium point
- Types of stability of linear systems
    - The stability of a linear system is determined by the eigenvalues of the linear system
- Stability of nonlinear linearised systems
    - The stability of a nonlinear system can be analysed by linearising the system around an equilibrium point and investigating the eigenvalues
    - If all (real parts of) the eigenvalues are negative, the system is stable; if one is positive, the system is unstable; if zero, undecided
- Liapunov criteria for stability
    - If there exists a Liapunov function around an equilibrium point of a nonlinear system, then the system is stable
- Importance of potential energy
    - The stability of a nonlinear dynamic system can be determined solely by the positive definiteness of the potential energy