AE4536: Buckling of structures

Discrete Multiple Degree-of-Freedom Structure

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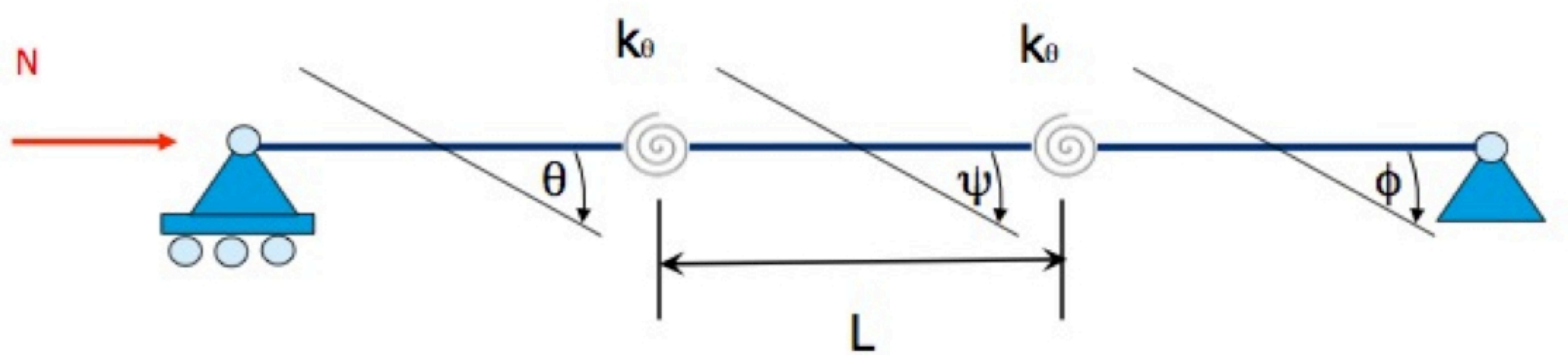
Problem definition

$$P(\theta,\psi,\varphi)=\frac{1}{2}k_\theta(\psi-\theta)^2+\frac{1}{2}k_\theta(\varphi-\psi)^2-N\delta$$

$$\delta=L(1-\cos\theta)+L(1-\cos\psi)+L(1-\cos\varphi)$$

$$\theta+\psi+\varphi=0 \rightarrow \psi=-\theta-\varphi$$



Potential energy

$$1-\cos\theta = 1-\left(1-\frac{1}{2!}\theta^2+\frac{1}{4!}\theta^4-\dots\right)$$
$$=\frac{1}{2}\theta^2$$

$$P(\theta,\varphi)=\frac{1}{2}k_\theta(-2\theta-\varphi)^2+\frac{1}{2}k_\theta(2\varphi+\theta)^2$$
$$-NL\left(\frac{1}{2}\theta^2+\frac{1}{2}(-\theta-\varphi)^2+\frac{1}{2}\varphi^2\right)$$

$$=\frac{1}{2}k_\theta(4\theta^2+4\varphi\theta+\varphi^2+4\varphi^2+4\varphi\theta+\theta^2)$$
$$-NL(\theta^2+\varphi^2+\varphi\theta)$$
$$=\frac{1}{2}k_\theta(5\theta^2+8\varphi\theta+5\varphi^2)-NL(\theta^2+\varphi^2+\varphi\theta)$$

Equilibrium

$$P(\theta_0+\delta\theta,\varphi_0+\delta\varphi)=\underbrace{\left(\frac{\partial P}{\partial\theta}\right)_0}_{=0}\delta\theta+\underbrace{\left(\frac{\partial P}{\partial\varphi}\right)_0}_{=0}\delta\varphi+h.\cancel{o}^{\,0}.t.=0=\delta P$$

$$\left(\frac{\partial P}{\partial\theta}\right)_0=\frac{1}{\cancel{2}}k_0(\cancel{10}^{5}\theta_0+\cancel{8}^{4}\varphi_0)-NL(2\theta_0+\varphi_0)$$

$$\left(\frac{\partial P}{\partial\varphi}\right)_0=\frac{1}{\cancel{2}}k_0(\cancel{10}^{5}\varphi_0+\cancel{8}^{4}\theta_0)-NL(2\varphi_0+\theta_0)$$

$$\begin{Bmatrix}\theta_0\\ \varphi_0\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Stability of equilibrium

$$P(\theta_0+\theta_1,\ \varphi_0+\varphi_1)=\frac{1}{2!}\left[\left(\frac{\partial^2 P}{\partial\theta^2}\right)_0\theta_1^2+\left(\frac{\partial^2 P}{\partial\theta\partial\varphi}\right)_0\theta_1\varphi_1+\left(\frac{\partial^2 P}{\partial\varphi\partial\theta}\right)_0\varphi_1\theta_1\right.$$

$$\left.+\left(\frac{\partial^2 P}{\partial\varphi^2}\right)_0\varphi_1^2\right]$$

$$\{\theta_1\ \ \varphi_1\}\begin{bmatrix}\dfrac{\partial^2 P}{\partial\theta^2} & \dfrac{\partial^2 P}{\partial\theta\partial\varphi}\\[2ex] \dfrac{\partial^2 P}{\partial\varphi\partial\theta} & \dfrac{\partial^2 P}{\partial\varphi^2}\end{bmatrix}_0\begin{Bmatrix}\theta_1\\ \varphi_1\end{Bmatrix} > 0 \quad \forall \begin{Bmatrix}\theta_1\\ \varphi_1\end{Bmatrix}$$

$= 0$ critical

Stability of equilibrium

$$\left(\frac{\partial^2 P}{\partial \theta^2}\right)_0 = 5k_\theta - 2NL$$

$$\frac{\partial^2 P}{\partial \theta \partial \varphi} = 4k_\theta - NL$$

$$\left(\frac{\partial^2 P}{\partial \varphi^2}\right)_0 = 5k_\theta - 2NL$$

$$\frac{\partial^2 P}{\partial \varphi \partial \theta} = 4k_\theta - NL$$

$$\underline{\underline{K}} = \begin{bmatrix} 5k_\theta - 2NL & 4k_\theta - NL \\ 4k_\theta - NL & 5k_\theta - 2NL \end{bmatrix}$$

Positive definite matrix

$$\begin{Bmatrix} x & y \end{Bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} > 0$$

$$ax^2 + 2bxy + cy^2 > 0$$

Necessary $a > 0 \wedge c > 0$ plug in $\{1\ 0\}$ $\{0\ 1\}$

$$\underbrace{a}_{>0}\left(x + \frac{b}{a}y\right)^2 + \underbrace{\left(c - \frac{b^2}{a}\right)}_{>0} y^2 \rightarrow \begin{array}{l} a > 0 \\ ac > b^2 \\ \rightarrow \det > 0 \end{array}$$

Buckling loads and buckling modes

$$\begin{vmatrix} 5k_\theta - 2NL & 4k_\theta - NL \\ 4k_\theta - NL & 5k_\theta - 2NL \end{vmatrix} = 0$$

$$(5k_\theta - 2NL)^2 - (4k_\theta - NL)^2 = 0$$

$$25k_\theta^2 + 4N^2L^2 - 20k_\theta NL - 16k_\theta^2 - N^2L^2 + 8k_\theta NL = 0$$

$$3L^2N^2 - 12k_\theta LN + 9k_\theta^2 = 0$$

$$D = 144k_\theta^2L^2 - 108k_\theta^2L^2$$
$$= 36k_\theta^2L^2$$

$$N = \frac{12k_\theta L \pm 6k_\theta L}{6L^2} = \frac{k_\theta}{L} \vee 3\frac{k_\theta}{L}$$

Buckling loads and buckling modes

$$N_1 = \frac{k_\theta}{L}$$

$$\begin{Bmatrix} 5k_\theta & -2k_\theta & 4k_\theta & -k_\theta \end{Bmatrix} \begin{Bmatrix} \theta_1 \\ \varphi_1 \end{Bmatrix}_1 = 0$$

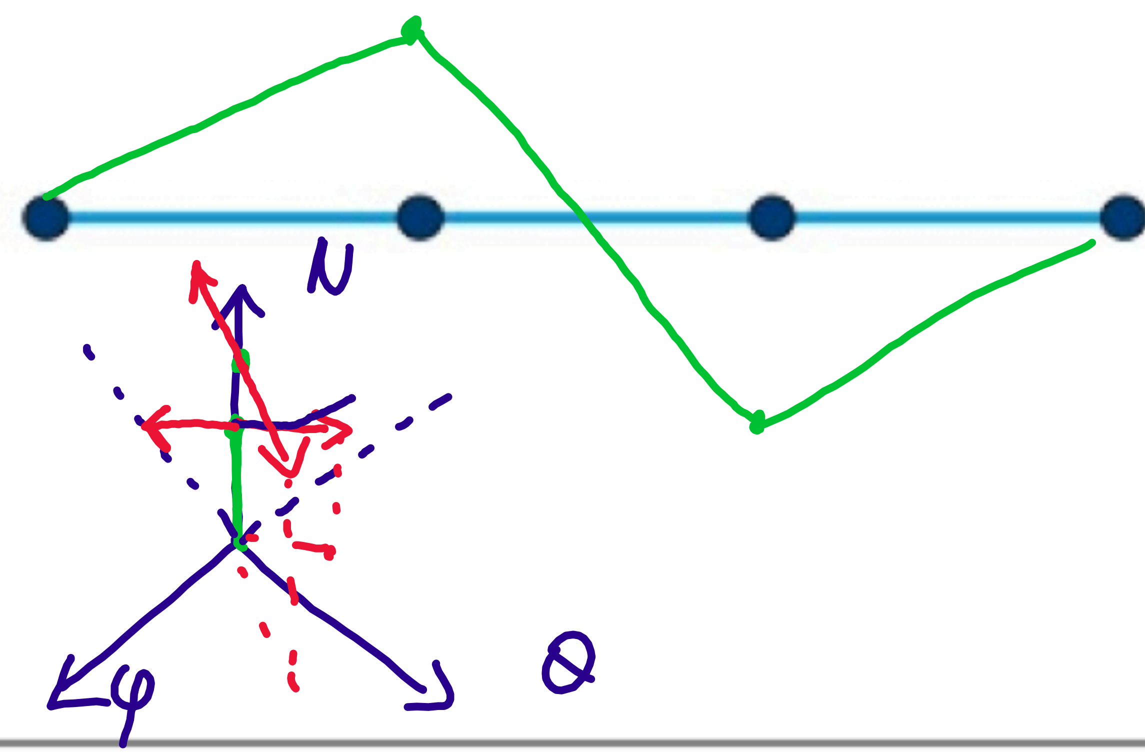
$$\theta_{11} + \varphi_{11} = 0 \qquad \begin{Bmatrix} \theta_1 \\ \varphi_1 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\psi_1 = 0$$

$$N_2 = \frac{3k_\theta}{L}$$

$$\begin{Bmatrix} -k_\theta & k_\theta \end{Bmatrix} \begin{Bmatrix} \theta_1 \\ \varphi_1 \end{Bmatrix}_2 = 0$$

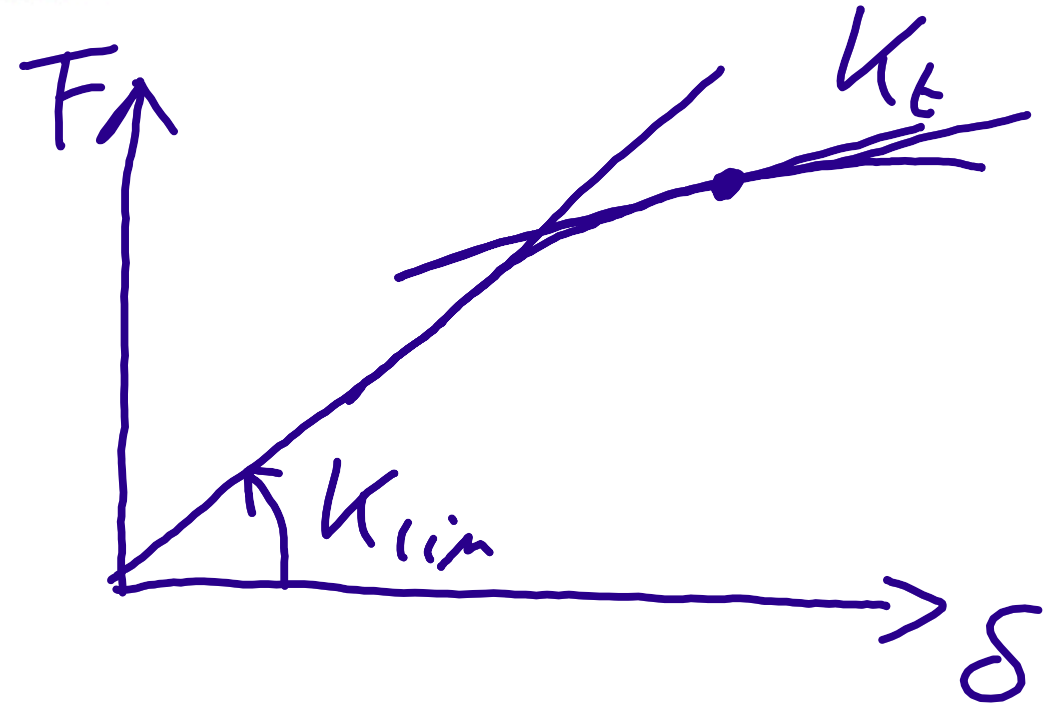
$$\begin{Bmatrix} \theta_1 \\ \varphi_1 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$



Geometric stiffness matrix

$$\underline{\underline{K}}_t = \underline{\underline{K}}_{lin} - N\,\underline{\underline{G}}$$

$$\underline{\underline{K}}_t = \begin{bmatrix} 5k_\theta & 4k_\theta \\ 4k_\theta & 5k_\theta \end{bmatrix} - N \begin{bmatrix} 2L & L \\ L & 2L \end{bmatrix}$$



eig. problem $\longrightarrow$ $$\left(\underline{\underline{K}}_{lin}\,\underline{\underline{G}}^{-1} - N\,\underline{\underline{I}}\right)\begin{Bmatrix} \theta \\ \varphi_1 \end{Bmatrix} = 0$$

Geometric stiffness matrix

$$\begin{bmatrix} 5k_\theta & 4k_\theta \\ 4k_\theta & 5k_\theta \end{bmatrix} \underbrace{\begin{bmatrix} 2L & -L \\ -L & 2L \end{bmatrix}}_{3L^2} = \begin{bmatrix} 2k_\theta/L & k_\theta/L \\ k_\theta/L & 2k_\theta/L \end{bmatrix}$$

$\xrightarrow{\text{eig. problem}}$

$$\begin{vmatrix} 2k_\theta/L - N & k_\theta/L \\ k_\theta/L & 2k_\theta/L - N \end{vmatrix} = 0$$

$$\longrightarrow \quad N = \frac{k_\theta}{L} \vee 3\frac{k_\theta}{L}$$

Summary

- Equilibrium and stability analysis of a multiple degree-of-freedom structure
- The calculation of critical loads and corresponding eigenmodes was shown
- The difference between the tangential, linear and geometric stiffness matrix was demonstrated