

# AE4536: Buckling of structures

## Rayleigh quotient

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## Rayleigh quotient for general structures

$$P(\underline{u}) = U(\underline{u}) - \lambda V(\underline{u})$$

$$\underline{u} = \sum_{i=1}^n a_i \varphi_i(x)$$

$$P(\underline{a}) = U(\underline{a}) - \lambda V(\underline{a})$$

$$\delta P(\underline{a}) = \frac{\partial U(\underline{a})}{\partial \underline{a}} - \lambda \frac{\partial V(\underline{a})}{\partial \underline{a}} = 0$$

$$\rightarrow \lambda_c = \frac{\partial U(\underline{a}) / \partial \underline{a}}{\partial V(\underline{a}) / \partial \underline{a}}$$

## Rayleigh quotient for beams



$$U = \frac{1}{2} \int_0^L EI \kappa^2 dx = \frac{1}{2} \int_0^L EI w''^2 dx = \frac{1}{2} \int_0^L EI a_i^2 \varphi_i''^2 dx$$

$$V = \frac{1}{2} P \int_0^L w'^2 dx = \frac{1}{2} P \int_0^L a_i^2 \varphi_i'^2 dx$$

$$\frac{\partial U}{\partial a_i} = a_i \int_0^L EI \varphi_i''^2 dx$$

$$\frac{\partial V}{\partial a_i} = a_i P \int_0^L \varphi_i'^2 dx$$

$$P_c = EI \frac{\int_0^L \varphi_i''^2 dx}{\int_0^L \varphi_i'^2 dx}$$

## Rayleigh quotient example


$$\varphi = x(L-x)$$

$$\varphi' = L - 2x$$

$$\varphi'' = -2$$

$$\varphi'^2 = L^2 - 4Lx + 4x^2$$

$$\varphi''^2 = 4$$


$$\int_0^L \varphi'^2 dx = L^3 - 2L^3 + \frac{4}{3}L^3 = \frac{1}{3}L^3$$
$$\int_0^L \varphi''^2 dx = 4L$$

$$P_c = EI \frac{4L}{\frac{1}{3}L^3} = \frac{12EI}{L^2} \approx 1,2 P_E \quad 20\% \text{ off}$$

## Timoshenko quotient

$$U = \frac{1}{2} \int_0^L EI \kappa^2 dx$$

$$M = EI \kappa$$

$$M^2/EI = EI \kappa^2$$

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$$= \frac{1}{2} \int_0^L \frac{M^2}{EI} dx$$


$$= \frac{1}{2} \int_0^L \frac{P^2 w^2}{EI} dx$$

$$P_c = EI \frac{\int_0^L \varphi'^2 dx}{\int_0^L \varphi^2 dx}$$



## Timoshenko quotient example


$$\begin{aligned}\varphi &= X(L-X) \\ \varphi^2 &= X^2(L^2 - 2LX + X^2) \\ &= X^4 - 2LX^3 + L^2X^2\end{aligned}$$



A diagram showing a horizontal blue line representing a beam of length L. A green parabolic curve is drawn above the beam, representing a deflection shape. The curve starts at the left end of the beam, rises to a peak in the middle, and returns to the right end of the beam. The length L is indicated by a horizontal line above the curve.

$$\begin{aligned}\int_0^L \varphi^2 dx &= \frac{1}{5}L^5 - \frac{1}{2}L^5 + \frac{1}{3}L^5 \\ &= \frac{1}{30}L^5\end{aligned}$$

$$P_c = EI \frac{\frac{1}{3}L^3}{\frac{1}{30}L^5} = \frac{10EI}{L^2} \approx 1,02 P_E \quad 2\% \text{ off}$$



# Summary

- The Rayleigh quotient can be used to calculate the approximate buckling load of a general structure
- The Timoshenko quotient can be used to calculate the approximate buckling load of beams only
- For beams, the Timoshenko quotient is mostly more accurate than the Rayleigh quotient