AE4536: Buckling of structures

Stability of beams

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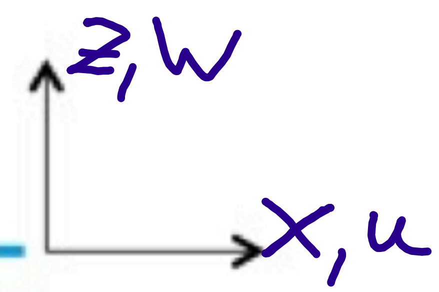
Beam formulation



Strain measures

$$\left.\begin{aligned}\varepsilon &= u' + \frac{1}{2} w'^2 \\ \kappa &= -w''\end{aligned}\right\} \rightarrow \underline{u} = \{u', w', w''\}$$

Potential energy

$$P = \frac{1}{2}\int_0^L EA\,\varepsilon^2\,dx + \frac{1}{2}\int_0^L EI\,\kappa^2\,dx = \frac{1}{2}\int_0^L N\varepsilon\,dx + \frac{1}{2}\int_0^L M\kappa\,dx$$

Potential energy expansion

$$P(\underline{u}_0 + \underline{u}_1) = \frac{1}{2}\int_0^L EA\left(u_0' + u_1' + \frac{1}{2}(w_0' + w_1')^2\right)^2 dx + \frac{1}{2}\int_0^L EI\,(w_0'' + w_1'')^2\,dx$$

Assemble terms of **u**1, (**u**$_1$,**u**$_1$), (**u**$_1$,**u**$_1$,**u**$_1$) etc to assess stability and equilibrium

Potential energy expansion

$$
\begin{aligned}
P(\mathbf{u}_0+\mathbf{u}_1) = {} & \frac{1}{2}EA\int_0^L \left(\cancel{u'^2_0}+u'^2_1+\cancel{\frac{1}{4}w'^4_0}+w'^2_0 w'^2_1+\frac{1}{4}w'^4_1+2u'_0 u'_1+\cancel{u'_0 w'^2_0}\right)dx \\
& +\frac{1}{2}EA\int_0^L \left(2u'_0 w'_0 w'_1+u'_0 w'^2_1+w'^2_0 u'_1+2w'_0 u'_1 w'_1+u'_1 w'^2_1\right)dx \\
& +\frac{1}{2}EA\int_0^L \left(w'^2_0 w'_0 w'_1+\frac{1}{2}w'^2_0 w'^2_1+w'_0 w'_1 w'^2_1\right)dx \\
& +\frac{1}{2}EI\int_0^L \left(w''^2_0+2w''_0 w''_1+w''^2_1\right)dx
\end{aligned}
$$

First form of potential energy

$$P(\mathbf{u}_0+\mathbf{u}_1)=\frac{1}{2}EA\int_0^L\left(u'^2_0+u'^2_1+\frac{1}{4}w'^4_0+w'^2_0w'^2_1+\frac{1}{4}w'^4_1\underline{+2u'_0u'_1}+u'_0w'^2_0\right)dx$$

$$+\frac{1}{2}EA\int_0^L\left(\underline{2u'_0w'_0w'_1}+u'_0w'^2_1\underline{+w'^2_0u'_1}+2w'_0u'_1w'_1+u'_1w'^2_1\right)dx$$

$$+\frac{1}{2}EA\int_0^L\left(\underline{w'^2_0w'_0w'_1}+\frac{1}{2}w'^2_0w'^2_1+w'_0w'_1w'^2_1\right)dx$$

$$+\frac{1}{2}EI\int_0^L\left(w''^2_0\underline{+2w''_0w''_1}+w''^2_1\right)dx$$

$$P_0'(\underline{u}_1)=\cancel{\tfrac{1}{2}}\int_0^L \underbrace{EA(\overbrace{2u_0'+w_0'^2}^{\varepsilon})}_{N}u_1'+\underbrace{EA(\overbrace{2u_0'+w_0'^2}^{\varepsilon})}_{N}w_0'w_1'+\underbrace{EI2w_0''}_{M}w_1'')\,dx$$

Equilibrium equations

$$P_0'(\delta \underline{u}) = \int_0^L \left(N\delta u_1' + N w_0' \delta w_1' + M\delta w_1''\right) dx$$

$$= \int_0^L \Big[(N\delta u)' - N'\delta u + (N w_0' \delta w)' - N' w_0' \delta w - N w_0'' \delta w + (M\delta w')' - (M'\delta w)' + M''\delta w \Big]\, dx = 0$$

	Equilibrium	Boundary terms
δu	$N' = 0$	$N\big\|_0^L = -P$
δw	$M'' - N w_0'' = 0$ $\underline{w_0 = 0}$ EIw_0''''	$N w_0' - M'\big\|_0^L = R_{0,L}$
$\delta w'$	/	$M\big\|_0^L = M_{ext}$

Partial integration

$$① \; (N\delta u)' = N'\delta u + N\delta u' \rightarrow N\delta u' = (N\delta u)' - N'\delta u$$

$$② \; (N w_0' \delta w)' = N' w_0' \delta w + N w_0'' \delta w + N w_0' \delta w'$$

$$\rightarrow N w_0' \delta w' = (N w_0' \delta w)' - N' w_0' \delta w - N w_0'' \delta w$$

$$③ \; (M\delta w')' = M'\delta w' + M\delta w'' \rightarrow M\delta w'' = (M\delta w')' - M'\delta w'$$

$$(M'\delta w)' = M''\delta w + M'\delta w' \rightarrow M'\delta w' = (M'\delta w)' - M''\delta w$$

$$\rightarrow M\delta w'' = (M\delta w')' - (M'\delta w)' + M''\delta w$$

Second form of potential energy

$$P(\mathbf{u}_0+\mathbf{u}_1)=\frac{1}{2}EA\int_0^L\left(u'^2_0+u'^2_1+\frac{1}{4}w'^4_0+w'^2_0w'^2_1+\frac{1}{4}w'^4_1+2u'_0u'_1+u'_0w'^2_0\right)dx$$

$$+\frac{1}{2}EA\int_0^L\left(2u'_0w'_0w'_1+u'_0w'^2_1+w'^2_0u'_1+2w'_0u'_1w'_1+u'_1w'^2_1\right)dx$$

$$+\frac{1}{2}EA\int_0^L\left(w'^2_0w'_0w'_1+\frac{1}{2}w'^2_0w'^2_1+w'_0w'_1w'^2_1\right)dx$$

$$+\frac{1}{2}EI\int_0^L\left(w''^2_0+2w''_0w''_1+w''^2_1\right)dx$$

$$P''_0(\underline{u}_1,\underline{u}_1)=\frac{1}{2}\int_0^L\left[EA\left(u'^2_1+w'^2_0w'^2_1+u'_0w'^2_1+2w'_0u'_1w'_1+\frac{1}{2}w'^2_0w'^2_1\right)+EI\,w''^2_1\right]dx$$

Stability equations

$$P_0''(\underline{u}_1,\underline{u}_1)=\frac{1}{2}\int_0^L\Big[\overbrace{EA\big(u_0'+\tfrac{1}{2}w_0'^2\big)}^{N_0}w_1'^2+EA\big(u_1'^2+\cancel{w_0'^2}\,w_1'^2+2\cancel{w_0'}\,u_1'w_1'\big)+EI\,w_1''^2\Big]dx$$

$$=\frac{1}{2}\int_0^L\Big[N_0w_1'^2+EA\,u_1'^2+EI\,w_1''^2\Big]dx$$

$$P_0''(\underline{u}_c,\delta\underline{u}_c)=\int_0^L\Big[N_0\,w_c'\,\delta w_c'+EA\,u_c'\,\delta u_c'+\underbrace{EI\,w_c''}_{M_c}\,\delta w_c''\Big]dx$$

Trefftz

Buckling solution

$$\begin{cases} EIM_c'' - N_0 W_c'' = 0 \\ u_c'' = 0 \end{cases}$$

$$N_0 W_c' - M_c \Big|_0^L = 0$$

$$u_c' \Big|_0^L = 0$$

$$M_c \Big|_0^L = 0$$

$$\begin{cases} W_c = A \sin(kx) + B \cos(kx) + Cx + D \\ u_c = \text{Constant} \end{cases}$$

$$k = \sqrt{\frac{P}{EI}}$$

$$EIW_c'''' + P W_c'' = 0$$

$$W_c'''' + k^2 W_c'' = 0$$

Third form of potential energy

$$P(\mathbf{u}_0+\mathbf{u}_1)=\frac{1}{2}EA\int_0^L\left(u'^2_0+u'^2_1+\frac{1}{4}w'^4_0+w'^2_0w'^2_1+\frac{1}{4}w'^4_1+2u'_0u'_1+u'_0w'^2_0\right)dx$$

$$+\frac{1}{2}EA\int_0^L\left(2u'_0w'_0w'_1+u'_0w'^2_1+w'^2_0u'_1+2w'_0u'_1w'_1+\underline{u'_1w'^2_1}\right)dx$$

$$+\frac{1}{2}EA\int_0^L\left(w'^2_0w'_0w'_1+\frac{1}{2}w'^2_0w'^2_1+\underline{w'_0w'_1w'^2_1}\right)dx$$

$$+\frac{1}{2}EI\int_0^L\left(w''^2_0+2w''_0w''_1+w''^2_1\right)dx$$

$$P_0'''(\underline{u}_c,\underline{u}_c,\underline{u}_c)=\frac{1}{2}\int_0^L EA\left(\cancel{u'_c}^{\,0}w_c'^2+w_0'\cancel{w'_c}^{\,0}w_c'^2\right)dx=0.\ \checkmark$$

Fourth form of potential energy

$$P(\mathbf{u}_0+\mathbf{u}_1)=\frac{1}{2}EA\int_0^L\left(u'^2_0+u'^2_1+\frac{1}{4}w'^4_0+w'^2_0w'^2_1+\underline{\frac{1}{4}w'^4_1}+2u'_0u'_1+u'_0w'^2_0\right)dx$$

$$+\frac{1}{2}EA\int_0^L\left(2u'_0w'_0w'_1+u'_0w'^2_1+w'^2_0u'_1+2w'_0u'_1w'_1+u'_1w'^2_1\right)dx$$

$$+\frac{1}{2}EA\int_0^L\left(w'^2_0w'_0w'_1+\frac{1}{2}w'^2_0w'^2_1+w'_0w'_1w'^2_1\right)dx$$

$$+\frac{1}{2}EI\int_0^L\left(w''^2_0+2w''_0w''_1+w''^2_1\right)dx$$

$$P_0''''(\underline{u}_c,\underline{u}_c,\underline{u}_c,\underline{u}_c)=\frac{1}{2}\int_0^L EA\,\frac{1}{4}w_c'^4 \geq 0$$

Stable

Summary

- The equilibrium and stability equation for an Euler beam was derived
- It was demonstrated that the equilibrium equation is nonlinear
- It was demonstrated that the stability equation is linear
- A beam has a stable critical point, which indicates that the post-buckling behaviour is expected to be stable