AE4536: Buckling of structures

Stability of beams under various boundary conditions

Roeland De Breuker
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Beam equations

$$EI\,w_0'''' - N\,w_0'' = 0 \qquad \text{Equilibrium}$$

$$EI\,w_c'''' + P_c\,w_c'' = 0 \qquad \text{Stability}$$

$$w_c'''' + k^2 w_c'' = 0$$

$$\begin{cases} w_c = A\sin(kx) + B\cos(kx) + Cx + D & \text{Buckling mode} \\ k^2 = \dfrac{P_c}{EI} \longrightarrow P_c = k^2 EI & \text{Buckling load} \end{cases}$$

Simply supported

NBCs $w''(0)=0$ $w''(L)=0$

EBCs $w(0)=0$ $w(L)=0$

Simply supported solution

$$w = A\sin(kx) + B\cos(kx) + Cx + D$$

$$w' = Ak\cos(kx) - Bk\sin(kx) + C$$

$$w'' = -Ak^2\sin(kx) - Bk^2\cos(kx)$$

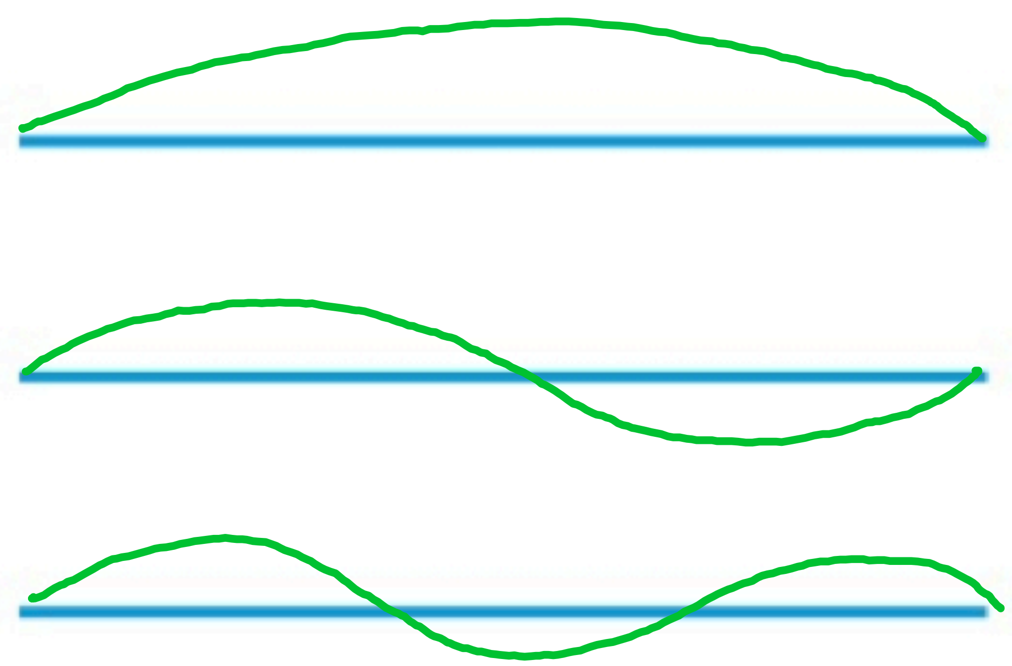
$$w''' = -Ak^3\cos(kx) + Bk^3\sin(kx)$$

1. $w''(0) = 0 = B$
2. $w(0) = 0 = B + D \rightarrow D = 0$
3. $w''(L) = 0 = -Ak^2\sin(kL) \rightarrow kL = n\pi$
4. $w(L) = 0 = C$

Simply supported solution

$$w_c = A \sin\left(\frac{n\pi x}{L}\right)$$

$$P_c = n^2 \frac{\pi^2 EI}{L^2} = P_E$$



$n=1$ — Symm

$n=2$ — Anti-sym

$n=3$ — Symm

Two side clamped

EBCs $w(0)=0$ $\quad$ $w(L)=0$

$w'(0)=0$ $\quad$ $w'(L)=0$

Two side clamped solution

$$w = A\sin(kx) + B\cos(kx) + Cx + D$$

$$w' = Ak\cos(kx) - Bk\sin(kx) + C$$

$$w'' = -Ak^2\sin(kx) - Bk^2\cos(kx)$$

$$w''' = -Ak^3\cos(kx) + Bk^3\sin(kx)$$

① $W(0) = 0 = B + D \rightarrow B = -D$

② $W'(0) = 0 = Ak + C \rightarrow C = -Ak$

③ $W(L) = 0 = A\sin(kL) + B\cos(kL) + CL + D$

④ $W'(L) = 0 = Ak\cos(kL) - Bk\sin(kL) + C$

Two side clamped solution

$$\begin{bmatrix} \sin(kL)-kL & \cos(kL)-1 \\ k(\cos(kL)-1) & -k\sin(kL) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} \rightarrow \left.\begin{matrix} \cos(kL)=1 \\ \sin(kL)=0 \end{matrix}\right\} kL = 2n\pi$$

$$\rightarrow A = C = 0$$

$$W_c = B(\cos kx - 1) = B\left(\cos\left(\frac{2n\pi x}{L}\right) - 1\right)$$

$$P_c = 4\,\frac{n^2\pi^2 EI}{L^2} = 4\,P_E$$

$n = 1$

Single side clamped

NBCs: $w''(L)=0$, $EIw'''(L)+P_c w'(L)=0$

EBCs: $w(0)=0$, $w'(0)=0$

Single side clamped solution

$$w = A\sin(kx) + B\cos(kx) + Cx + D$$

$$w' = Ak\cos(kx) - Bk\sin(kx) + C$$

$$w'' = -Ak^2\sin(kx) - Bk^2\cos(kx)$$

$$w''' = -Ak^3\cos(kx) + Bk^3\sin(kx)$$

(1) $EIw'''(L) + P_c w'(L) = 0 = C$

(2) $w'(0) = 0 = Ak + C \rightarrow A = 0$

(3) $w''(L) = 0 = B\cos(kL) = 0 \rightarrow kL = \frac{2n-1}{2}\pi$

(4) $w(0) = 0 = B + D \rightarrow B = -D$

Single side clamped solution

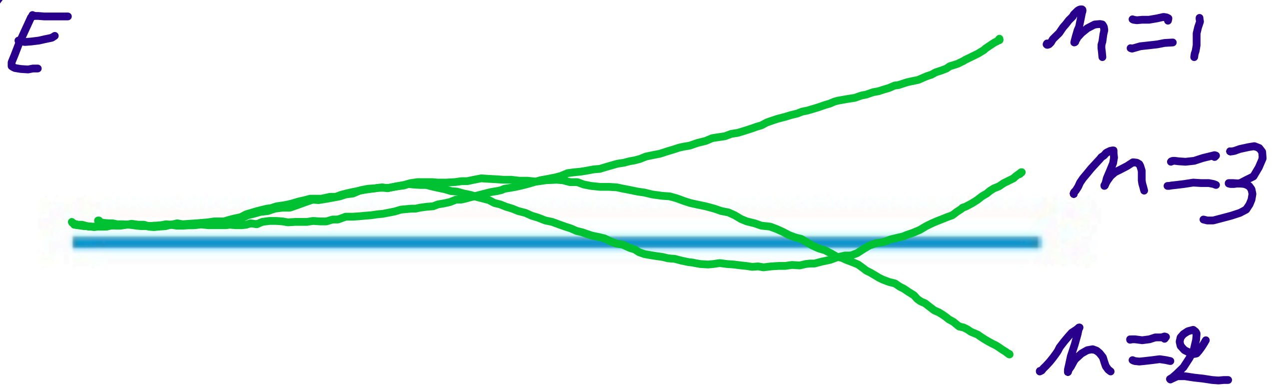
$$W_c = B\left(\cos\left(\frac{2n-1}{2}\,\frac{\pi x}{L}\right)-1\right)$$

$$P_c = \left(\frac{2n-1}{2}\right)^2 \frac{\pi^2 EI}{L^2}$$

$n=1$

$$W_{c,1} = B\left(\cos\left(\frac{\pi x}{2L}\right)-1\right)$$

$$P_{c,1} = \frac{1}{4}\,\frac{\pi^2 EI}{L^2} = \frac{1}{4}P_E$$



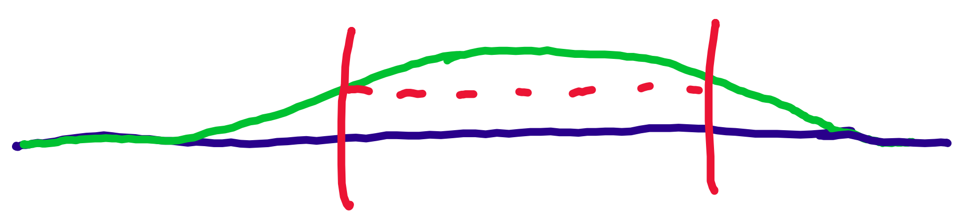
Effective length

$$P_{c,1} = c\,\frac{\pi^2 EI}{L^2}$$

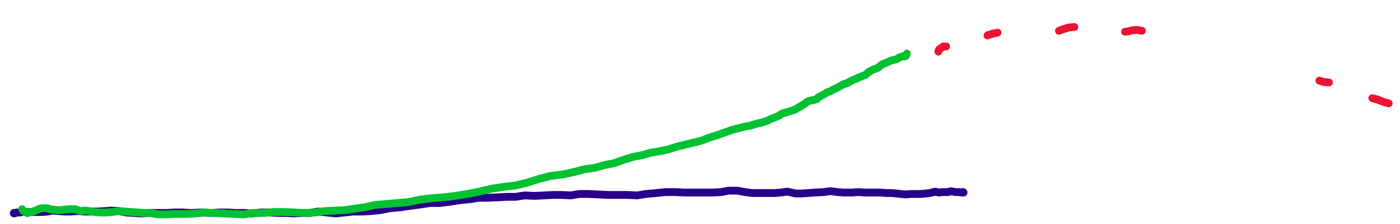
$$L_e^2 = \frac{L^2}{c} \rightarrow P_E = \frac{\pi^2 EI}{L_e^2}$$

	c	L_e
SS	1	L
CC	4	$\frac{1}{2}L$
CF	$\frac{1}{4}$	$2L$

CC



CF



Summary

- Buckling loads and modes have been calculated by solving the governing beam stability equation under various boundary conditions
- The Euler buckling load has been derived
- The effect of boundary conditions on the buckling load and mode has been demonstrated