Chapter 25.
The Core

- Imputation
- Domination
- Stable Sets

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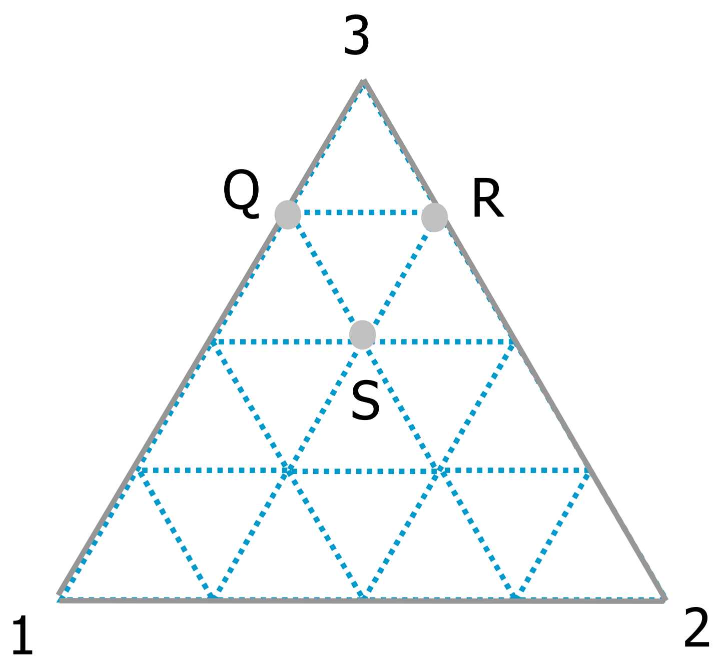
Triangle Diagrams



- Imputations from three-player games may be represented on a triangle
- Imputations closer to a player result in that player receiving a larger fraction of the total
- The value of the supercoalition is taken as a constant
- Examples:

Q {0.25, 0.00, 0.75}
R {0.00, 0.25, 0.75}
S {0.25, 0.25, 0.50}

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Strategic Equivalence of Characteristic Functions

- Recall that we can measure utility for an individual only on an interval scale
- Recall that we cannot compare utility between players
- Thus, any individuals utility is non-unique subject to order preserving transformations
- Straffin suggests we convert our games using a 0-1 normalization
  - Lone players get zero
  - Supercoalition gets one

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Some Terminology

- Additive: $v(S) + v(T) = v(S \cup T)$

  Games must be superadditive if coalitions are to form
- Zero Sum: $v(S) + v(N \setminus S) = v(N)$
- Essential $v(N) > \sum_{i \in N} v(\{i\})$

Individual Rationality

- **In analytical form:** $x_i \geq \nu(i)$ for all i
- **In words:** Each player should receive at least as much as they would have received if they had gone it alone
- **A defense:** If a player receives more than they are assigned by dropping out of collective arrangements, then of course they will drop out of the group. The group as a whole is likely to suffer.

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Collective Rationality

- **In analytical form:** $$\sum_{i \in N} x_i = \nu(N)$$
- **In words:** We should distribute the value of the grand coalition amongst all members.
- **A defense:** We can't gain more value than the grand coalition, and our imputation would be inefficient if we divided less than what we had available.

Alternative Requirements for a Solution

- Known as the Core
- Gillies and Shapley (c. 1955)
- Individual rationality
- Collective rationality, also known as group rationality
- Suitable for non-zero sum cooperative games

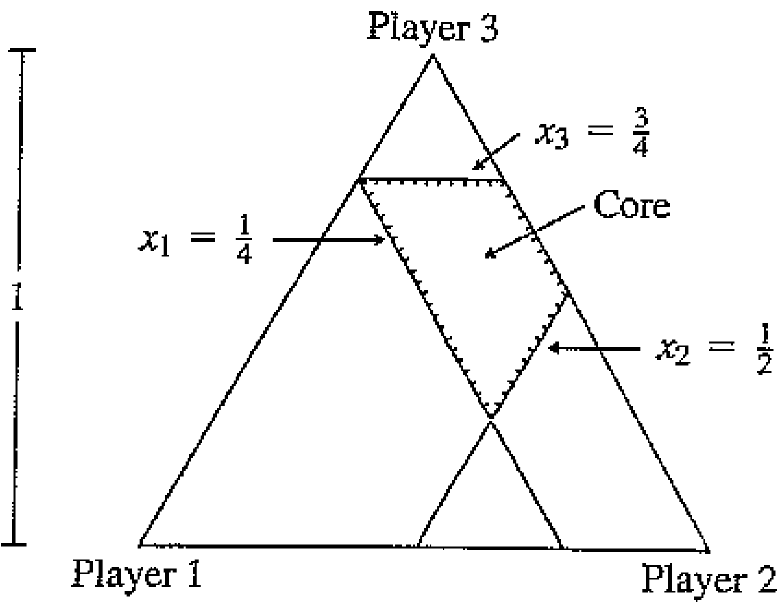
Definition of the Core

- The *core* of a game in characteristic function form is the set of all undominated imputations, i.e., the set of all imputations (x_1, ..., x_n) such that:

$$\textit{for all } S \subseteq N, \sum_{i \in S} x_i \geqslant v(S).$$

- In short, the core are all solutions obeying individual and group rationality

Example 1: Straffin's Game



$$v(1) = v(2) = v(3) = 0$$

$$v(12) = \frac{1}{4} \qquad v(13) = \frac{1}{2} \qquad v(23) = \frac{3}{4}$$

$$v(123) = 1$$

Diagram and example from Game Theory and Strategy (Straffin 1993) p.166

Linear Feasibility Problem

$$v(1) = v(2) = v(3) = 0$$
$$v(12) = \tfrac{1}{4} \qquad v(13) = \tfrac{1}{2} \qquad v(23) = \tfrac{3}{4}$$
$$v(123) = 1$$

An imputation (x_1, x_2, x_3) is in the core of this game if

$$x_1 + x_2 \geqslant v(12) = \tfrac{1}{4}$$
$$x_1 + x_3 \geqslant v(13) = \tfrac{1}{2}$$
$$x_2 + x_3 \geqslant v(23) = \tfrac{3}{4}.$$

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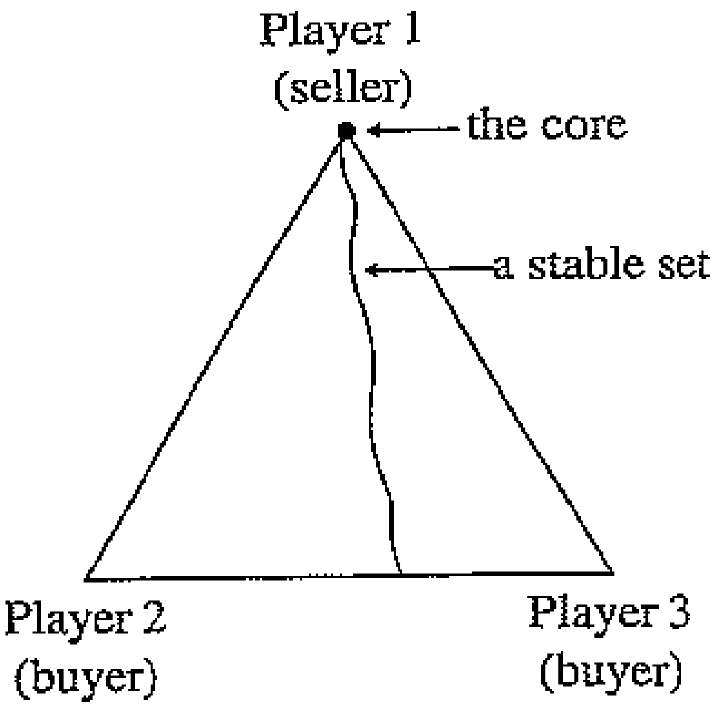
Example 2: House Buying Game



$$v(1) = v(2) = v(3) = 0$$

$$v(12) = v(13) = 1 \qquad v(23) = 0$$

$$v(123) = 1$$

Diagram from Game Theory and Strategy (Straffin 1993) p.167

Core and Stable Set: Contrast

- The **core** captures the effect of competition
- The **stable set** captures the effect of collusion
- There is no reason why the core should be a stable set
- The core may be dominated by all other available imputations!