Chapter 26.
The Shapley Value

- Understanding and Conceptualizing the Value
- Calculating the Value
- Other Related Models

Permutations and Combinations

Combination

Choosing things when ordering does not matter. Choosing n things from a pool of k is represented by

$\binom{n}{k}$ and is calculated by $\frac{n!}{(n-k)!k!}$

The Excel formula is COMBIN(n, k)

Permutation

Choosing things when ordering matters. Choosing n things is represented by

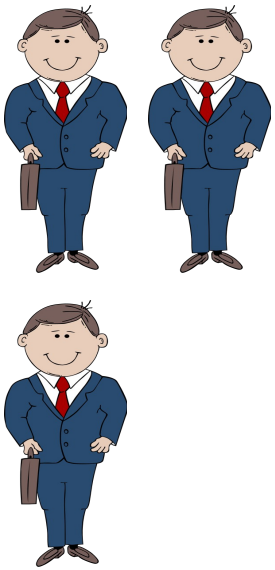
n! and is calculated by 1*2*…(n-1)*n

The Excel formula is FACT(n)

What is the Shapley Value

- A single imputation for a multi-player game
- An attempt to provide a fair solution to multi-player games, which previously had not existed
- A division of the proceeds of the grand coalition
- Note that the Shapley value may not be individual or group rational; it may need to be imposed by the decision-maker
- Applies to cost sharing, as well as value sharing!

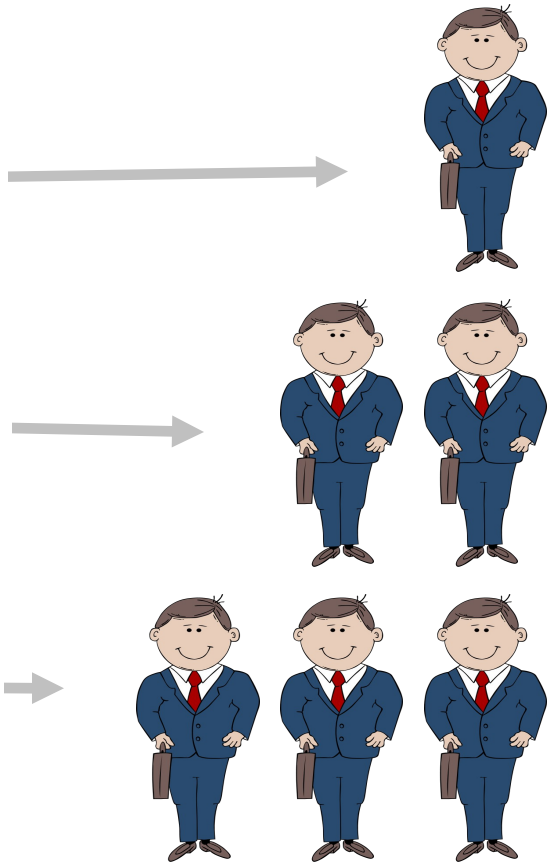
Understanding the Shapley Value



Player one starts a coalition, securing their own value

Player two joins the coalition, gaining all of the marginal value from the coalition of one and two.

Player three joins the coalition, gaining all the marginal value.



Notes on the Conceptualization

- Some players
  - introduce negative marginal values and are penalized accordingly.
  - never create marginal value no matter who they join and are penalized accordingly.
- Shapley value rewards those players who repeatedly create high values by joining coalitions, regardless of the order in which they arrive.

Example

Order	A	B	C
ABC	0	2	5
ACB	0	3	4
BAC	2	0	5
BCA	1	0	6
CAB	4	3	0
CBA	1	6	0
Sum	8	14	20
Ave	8/6	14/6	20/6

$$\nu(A)=\nu(B)=\nu(C)=0$$
$$\nu(AB)=2 \quad \nu(AC)=4 \quad \nu(AC)=4$$
$$\nu(ABC)=7$$

Example from Game Theory and Strategy (Straffin 1993) p.173

Pitfalls in Calculating Shapley Value

Order	A	B	C
ABC	0 (1)	2 (2)	5 (3)
ACB	0 (1)	3 (3)	4 (2)
BAC	2 (2)	0 (1)	5 (3)
BCA	1	0	6
CAB	4	3	0
CBA	1	6	0
Sum	8	14	20
Ave	8/6	14/6	20/6

Fill values by the order of arrival, not by the ordering of the columns

$$\nu(A)=\nu(B)=\nu(C)=0$$
$$\nu(AB)=2 \quad \nu(AC)=4 \quad \nu(AC)=4$$
$$\nu(ABC)=7$$

All values are marginal.

TU Delft

Other Ways of Calculating

Coalition S	(s-1)!(n-s)!	$\nu(S)-\nu(S-i)$	Product
A	1x6=6	0 – 0 = 0	0
AB, etc.	1x2=2	50 - 0 = 50	100

↑ Focus on a single player to reduce needed calculations. Who was in the forming grand coalition before player A arrived?

↑ Permutation calculations. Note that this is a four player game.

↑ Marginal value calculations. How much value does the total coalition create? How much could it have created without player A?

↑ Calculate the product. This is used in a weighted average.

Table adapted from Game Theory and Strategy (Straffin 1993) p.175

Other Ways of Calculating

Coalition S	(s-1)!(n-s)!	$\nu(S)-\nu(S-i)$	Product
A	1x6=6	0 – 0 = 0	0
AB	1x2=2	50 - 0 = 50	100
AC	1x2=2	30 – 0 = 30	60
AD	1x2=2	30 – 0 = 30	60
ABC	2x1=2	120 – 90 = 30	60
ABD	2x1=2	120 – 90 = 30	60
ACD	2x1=2	120 – 70 = 50	100
ABCD	6x1=6	120 – 120 = 0	0
	24	Shapley Value	440/24

Table from Game Theory and Strategy (Straffin 1993) p.175

Symmetries and Short Cuts

- Indistinguishable players always get the same value
- Names don't matter; coalition structure does
- Thus you can calculate the Shapley value once, and then give this value to all players
- Also, the Shapley value is Pareto optimal
- So for instance in a three-player game, if you have already calculated the Shapley value for players one and two, you know what the value of three must be

Myerson Value

- An extension of Shapley
- Suppose players exist in a network.
- Suppose A cannot join a coalition with C unless they are introduced by B, and vice versa.
- Certain orderings are prohibited.
- In this game, having B as the broker improves the outcomes of player C.

Order	A	B	C
ABC	0	2	5
~~ACB~~	~~0~~	~~3~~	~~4~~
BAC	2	0	5
BCA	1	0	6
~~CAB~~	~~4~~	~~3~~	~~0~~
CBA	1	6	0
Myerson	4/4	8/4	16/4
Shapley	8/6	14/6	20/6

Comparing and Contrasting Methods

- Number of Players

- Prescriptive: Choose what's fair

vs

- Descriptive: Describing what is likely given power or other arrangements

	Two-player	N-player
Prescriptive	Nash Arbitration	Shapley Value
Descriptive	Asymmetric Nash, Rubinstein Bargaining	Stable Set, Core