

Chapter 31. The Nucleolus and Gately Point

- Nucleolus
- Gately Point
- Optimization Solutions

In Brief



- Grease the most squeaky wheel until it squeaks as little as possible
- The Nucleolus treats all coalitions as equally important
- The Gately point attempts to secure the most valuable coalition members as best as possible

U.S. Army photo by Army Pfc. Eric Liesse (public domain, usgov)

Nucleolus and Gately variants

- The Nucleolus treats all coalitions as equally important (Schmeidler, 1970)
- The Gately point attempts to secure the most valuable coalition members as best as possible (Gately, 1974)
- Both variants of the “squeaky wheel”

The Excess

- Define the **excess** as

$$e_S(\mathbf{x}) = v(S) - \sum_{i \in S} x_i$$

- The excess is always specified for a given coalition S

Interpreting the Excess

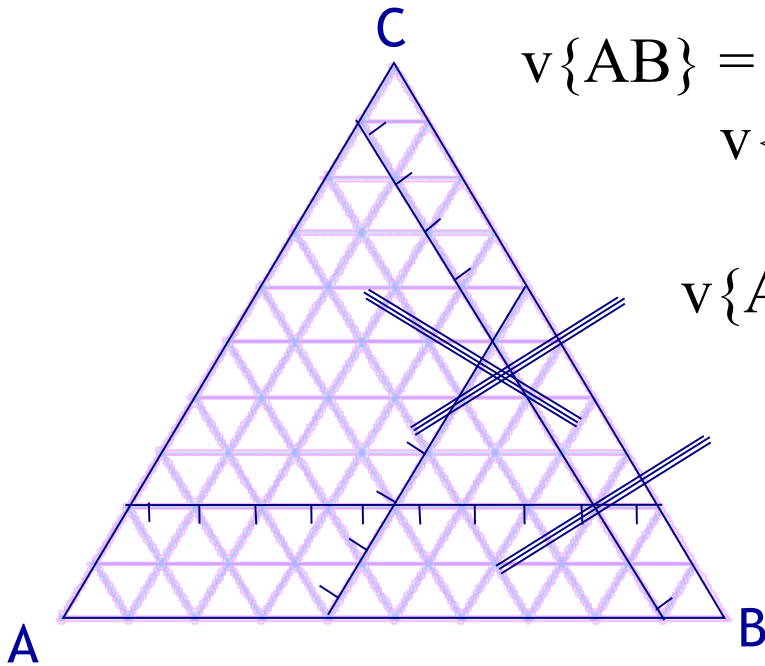
- The excess is defined as the difference between what the coalition could earn alone, and the amount they are awarded by others
- As defined, a positive quantity when underfunded, and a negative quantity when overfunded
- Interpretable as “the unhappiness” of the coalition

A Graphical Solution

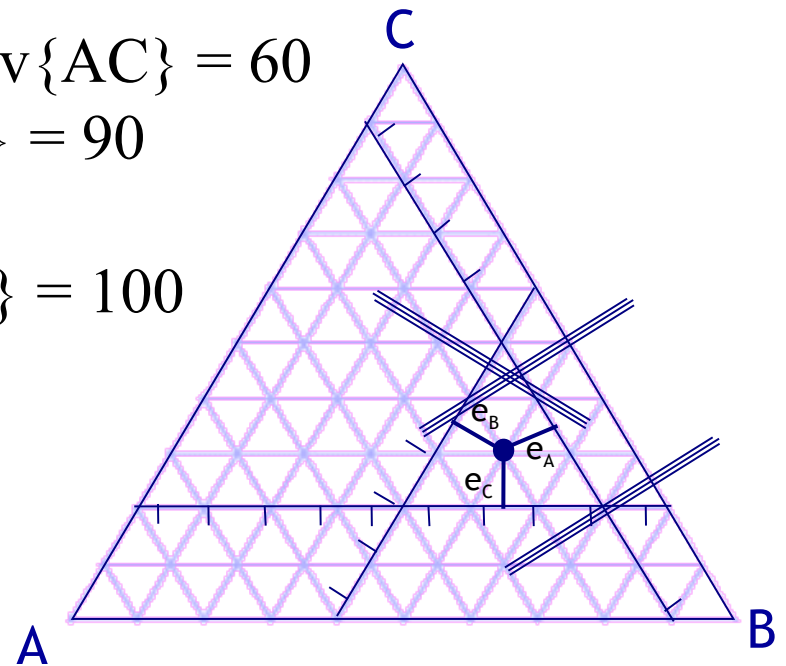
$$v\{A\} = v\{B\} = v\{C\} = 0$$

$$v\{AB\} = 80 \quad v\{AC\} = 60 \\ v\{BC\} = 90$$

$$v\{ABC\} = 100$$



An Empty Core



The Nucleolus

Solution Concepts

- Guided Search:
 - Set the excess of the grand coalition to zero.
 - Then minimize the next greatest value, keeping all other excesses constant.
 - Proceed down the list until no more improvements possible.
- Graphical / Geometrical: Often easiest when working with three-player games.

The Propensity to Disrupt

- Define the **propensity to disrupt** as

$$d_i(\mathbf{x}) = \frac{\sum_{j \neq i} x_j - \nu(N-i)}{x_i - \nu(i)}$$

- The propensity is always specified for a given player i
- May be extended to coalition S , in which case the solution is called the *disruption nucleolus*

Interpreting the Propensity to Disrupt

- “If I desert the grand coalition, I’ll lose something, but you’ll lose a lot more!”
- In words: propensity to disrupt is the ratio between what you lose and what I lose when I desert
- Thus I’m more likely to disrupt if I have nothing to lose, or if you are gaining a lot from my cooperation

Solving for the Gately Point

- In order to minimize disruption you must find an imputation so that the propensity for disruption is equal for all players
- The required payoff for all players is then proportional to

$$v(N) - v(N - i)$$

- This is the marginal value of player i to the supercoalition N

A Graphical Comparison

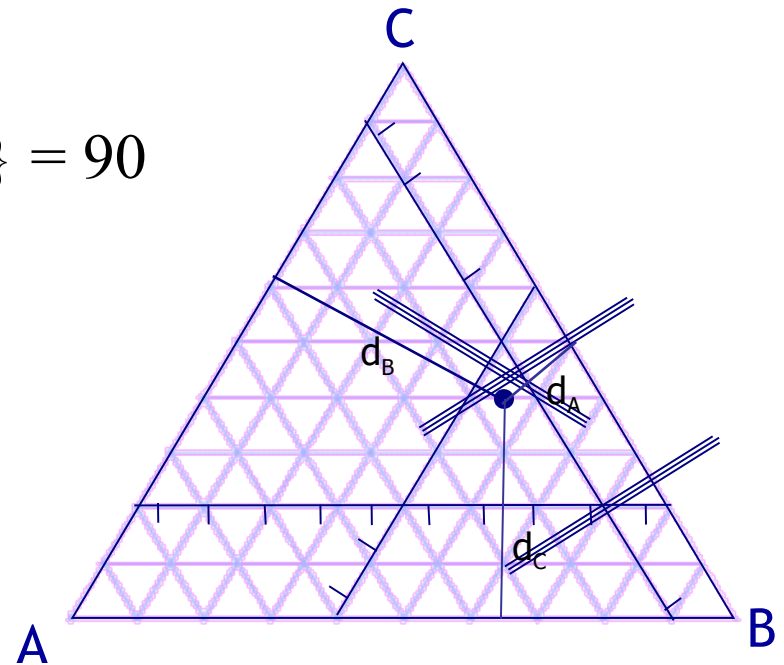
$$v\{A\} = v\{B\} = v\{C\} = 0$$

$$v\{AB\} = 80 \quad v\{AC\} = 60 \quad v\{BC\} = 90$$

$$v\{ABC\} = 100$$

Solution Proportional to Disruptive Capability

$$\begin{aligned} 100 - 90 &: 100 - 60 : 100 - 80 \\ &1:4:2 \\ &14:58:38 \end{aligned}$$



The Gately Point

Example Nucleolus Optimization

Microsoft Excel - CooperativeWorksheet.xls

File Edit View Insert Format Tools Data Window Help

Type a question for help

100% Arial 10

Security...

F23 =MAX(F7:F20)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1			Nucleolus Worksheet															
2																		
3																		
4																		
5																		
6			Coalition	Value	Payment from Imputation	Excess												
7			v{A}	0	28.6	-28.6		x_A	28.6									
8			v{B}	0	18.7	-18.7		x_B	18.7									
9			v{C}	0	9.7	-9.7		x_C	9.7									
10			v{D}	0	43.0	-43.0		x_D	43.0									
11			v{AB}	45	47.3	-2.3												
12			v{AC}	19	38.3	-19.3		Constraint	100									
13			v{AD}	2	71.6	-69.6												
14			v{BC}	26	28.4	-2.4												
15			v{BD}	28	61.7	-33.7												
16			v{CD}	24	52.7	-28.7												
17			v{BCD}	69	71.4	-2.4												
18			v{ACD}	79	81.3	-2.3												
19			v{ABD}	82	90.3	-8.3												
20			v{ABC}	54	57.0	-3.0												
21			v{ABCD}	100	100.0	0.0												
22																		
23					Objective:	-2.3												
24					Min													
25					Max													
26					Excess													
27																		

Solver Parameters

Set Target Cell:

Equal To: Max Min Value of:

By Changing Cells:

Subject to the Constraints:

Buttons: Solve, Close, Options, Add, Change, Delete, Reset All, Help