Optimization, a repetition :

The second topic (besides solving DE'-s) of these lectures is to optimize functions of several variables.
Question: What do we mean with this?

- Given again the function $f(x, y) = \sqrt{x^2 + y^2}$.
- Suppose we are "sitting" in point $P(1, 1)$ and do several steps with a given size in different directions.
- Because of these steps f is changing.
- Question: What is the most optimal step direction?
- Answer:If you take the distance interpretation for f, the answer is simple: Start at P and step in the direction $\langle 1, 1 \rangle$.
- An analog question for a solution function of a DE $x(v, m, c, k, t)$ is much more complicated!

Tools for Optimization: partial derivative

To answer the general optimization question, the following preliminary tools and answers on questions are needed :
Questions:

1. Question1:
   Given $f(x, y) = \sin(x^2 y^3)$.
   Find: $\frac{\partial \sin(x^2 y^3)}{\partial x}$
   Find: $\frac{\partial \sin(x^2 y^3)}{\partial y}$
2. Question2:
   Given a function $g(x, y)$ and a point $P(p_0, p_1)$.
   What is the meaning of $\frac{\partial g(P)}{\partial x} > 0$
   and $\frac{\partial g(P)}{\partial y} > 0$

Differentiation-rules:

The basic rules for differentiation:

1. *Summation rule*: $(f+g)' = f' + g'$
2. *Product rule*: $(f \cdot g)' = f' \cdot g + f \cdot g'$
3. *Quotient rule*: $(\frac{f}{g})' = \frac{f' \cdot g - f \cdot g'}{g^2}$
4. *Chain rule*: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

Some basic examples:

Find in each case the derivative:

1. *Summation rule*:
   $(x^2 + \sin(x))' = 2x + \cos(x)$.
2. *Product rule*:
   $(x^2 \cdot \sin(x))' = 2x \cdot \sin(x) + x^2 \cdot \cos(x)$.
3. *Quotient rule*:
   $(\frac{x^2}{\sin(x)})' = \frac{2x \cdot \sin(x) - x^2 \cdot \cos(x)}{\sin^2(x)}$.
4. *Chain rule*:
   $(\sin(x^2))' = \cos(x^2) \cdot 2x$.

The meaning of a derivative:

What is meaning of $f'(a) > 0$?

Some possible answers:

- Function f is increasing.
- The slope of the tangent-line is positive

The most exact answer is from the *definition* is:

$$\boxed{\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} > 0}$$

Try to understand this value of this answer, why is it better than former ones?

The meaning of a partial derivative:

Given a function of 2 variables $f(x,y)$ and a point $P(a,b)$
The partial derivative of $f(P)$ w.r.p. to x is:

$$\boxed{\lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}}$$

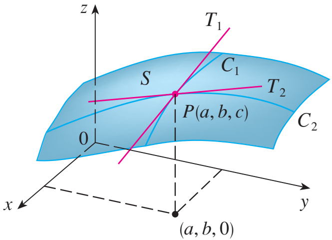
notation: $\frac{\partial f(P)}{\partial x}$.

The partial derivative of $f(P)$ w.r.p. to y is:

$$\boxed{\lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}}$$

notation: $\frac{\partial f(P)}{\partial y}$.

The meaning of a partial derivative:



Question: In the picture above; determine the sign of $\frac{\partial f(P)}{\partial x}$.
Answer: <0,
and the sign of $\frac{\partial f(P)}{\partial y}$. Answer: >0
Remark:
In this picture you see the two tangents belonging to the partial derivatives. (Both are member of the same tangent plane)

Determining a partial derivative:

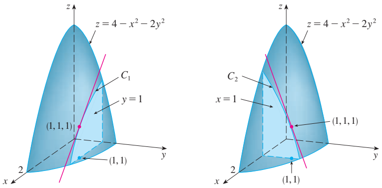
From these definitions we can deduce that finding the derivative of a function $f(x, y)$ is nothing more than ordinary differentiation whereby the other variable is taken constant:

Example 1: $\frac{\partial}{\partial x}(x^2y^3) = 2xy^3$.

Example 2: $\frac{\partial}{\partial y}(x^2y^3) = 3x^2y^2$.

Example 3: $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(x^2y^3)\right) = 6xy^2$. A brief notation is $\frac{\partial^2}{\partial x \partial y}(x^2y^3)$, a briefer $(x^2y^3)_{yx}$.

The meaning and calculation of a partial derivative:



Question: In the above pictures you see the graph of the function $f(x, y) = 4 - x^2 - 2y^2$ and the tangents at (1,1,1) belonging to the two partial derivatives. Find their slopes and are signs of slopes in accordance with the picture?

answer: $f_x(x, y) = -2x$, so $f_x(1, 1) = -2$ so the slope of the first tangent is -2, so the value of f at (1,1) is decreasing with increasing x!, which is seen in the picture.

$f_y(x, y) = -4y$, so $f_y(1, 1) = -4$ so the slope of the second tangent is -4, so the value of f at (1,1) is decreasing with increasing y!, which is also seen in the picture.

Determining partial derivatives of $f(x_1, x_2 \cdots, x_n)$:

Given $f(x_1, x_2, x_3, \cdots, x_n) = x_1^1 x_2^2 x_3^3 \cdots x_n^n$.

1. Find $\frac{\partial f}{\partial x_2}$
2. Given $P(1, 1, \cdots 1)$, We take at P a small step in (the positive) "x_2"-direction and a same small step in (the positive) "x_3"-direction. In which case is the increase of f the most? and if we do at P small steps in the opposite directions?

Further reading and exercises see Stewart 14.3

Vectors and Lines in $\mathbb{R}^3$, a repetition

Question1:

Given cube $\mathcal{C}$ with

$A(0,0,0), B(1,0,0), C(1,1,0), D(0,1,0), E(0,0,1), F(1,0,1), G(1,1,1), H(0,1,1)$

1. Find a vector equation for line l_1 through A and C.

   Answer: $l_1 = \lambda_1 \langle 1, 1, 0 \rangle$

2. Find a vector equation for line l_2 through E and C.

   Answer: $l_2 = \langle 0, 0, 1 \rangle + \lambda_2 \langle 1, 1, -1 \rangle$

Vectors and Lines in $\mathbb{R}^3$, a repetition

Question2:

1. Give the vector equation in $\mathbb{R}^2$ of the tangent (line) l of the graph "$y = x^2$" at point $P(1,1)$.

   Answer:$l = \langle 1, 1 \rangle + \lambda \langle 1, 2 \rangle$

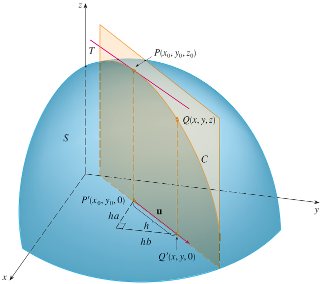
2. Given the graph of $f(x, y) = x^2 y^3$ and point $P(1, 1, 1)$ on it.
   Find the vector equations of tangent lines l_1 and l_2 of the graph at P in the plane $x = 1$ and in the plane $y = 1$.

   Answer:$l_1 = \langle 1, 1, 1 \rangle + \lambda \langle 0, 1, 3 \rangle$ and $l_2 = \langle 1, 1, 1 \rangle + \lambda \langle 1, 0, 2 \rangle$

Directional Derivative

In the last slide we were dealing with tangent lines at a point P in special directions namely in the x and y direction.

View the next picture:



In the picture you see a tangent line T in a random direction at $P(x_0, y_0, z_0)$, this direction is determined by a unit vector **u** which lies in xy-plane.

Directional Derivative

We want to calculate the slope, which we note as $D_{\mathbf{u}}f(x_0, y_0)$, of the tangent line T.

Definition of $D_{\mathbf{u}}f(x_0, y_0)$:

$$\boxed{D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0, y_0)}{h}}$$

hereby is $\mathbf{u} = \langle a, b\rangle$ with $a^2 + b^2 = 1$, *the unit direction*.

Question:
Can we calculate $D_{\mathbf{u}}f(x_0, y_0)$ without taking the limit?

Directional Derivative

Answer: In 4 steps:

1. Question1:
   Give the vector equation of the tangent line T
   Answer: $T = \langle x_0, y_0, f(x_0, y_0)\rangle + \lambda\langle a, b, D_{\mathbf{u}}f(x_0, y_0)\rangle$
2. Question2:
   Why lie the three space vectors $\mathbf{v} = \langle a, b, D_{\mathbf{u}}f(x_0, y_0)\rangle$, $\mathbf{v}_1 = \langle 1, 0, f_x(x_0, y_0)\rangle$ and $\mathbf{v}_2 = \langle 0, 1, f_y(x_0, y_0)\rangle$ in one plane?
   Answer: They are direction vectors of three tangent lines of the graph of f at point $P(x_0, y_0, f(x_0, y_0))$. so they lie all in the tangent plane
3. Question3:
   Write $\mathbf{v}$ as a (linear) combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
   Answer: $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$
4. Question4:
   Express $D_{\mathbf{u}}f(x_0, y_0)$ in $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$
   Answer: From last answer follows $D_{\mathbf{u}}f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0)$
5. So $\boxed{D_{\mathbf{u}}f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0)}$

Directional Derivative: an exercise

Back to our two dimensional distance function:

1. Given $f(x,y) = \sqrt{x^2+y^2}$ and $P(1,1)$. Calculate $D_{\mathbf{u}}f(P)$ for the unit-steps $\mathbf{u} = \langle 1,0\rangle, \mathbf{u} = \langle 0,1\rangle$ and $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

   Answer: $D_{\langle 1,0\rangle}f(P) = \frac{1}{\sqrt{2}}, D_{\langle 0,1\rangle}f(P) = \frac{1}{\sqrt{2}}, D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle}f(P) = 1$

2. Calculate $D_{\mathbf{u}}f(P)$ for the general unit step $\mathbf{u} = \langle \cos(\theta), \sin(\theta)\rangle$

   Answer: $D_{\langle \cos\theta, \sin\theta\rangle}f(P) = \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)$

3. For which θ in the last problem is $D_{\mathbf{u}}f(P)$ a maximum?

   Answer: $\theta = \frac{\pi}{4}$

For more exercises and theory see Stewart 14.7

Dot product:a Repetition

Given two vectors $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$
Do you remember:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

whereby θ is the angle between the given vectors.
What has this to do with:

$$D_{\mathbf{u}} f(x_0, y_0) = a f_x(x_0, y_0) + b f_y(x_0, y_0)$$

a convenient formula for $D_{\mathbf{u}}f(x_0, y_0)$

$$D_{\mathbf{u}}f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle \cdot \langle a, b\rangle.$$

and

$$D_{\mathbf{u}}f(x_0, y_0) = |\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle||\langle a, b\rangle| \cos(\theta).$$

In the last formula is θ angle between vectors $\mathbf{u} = \langle a, b\rangle$ and $\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$
Because $\mathbf{u} = 1$ (unit vector) we get

$$D_{\mathbf{u}}f(x_0, y_0) = |\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle| \cos(\theta)$$

We note $\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$ as $\nabla f(x_0, y_0)$ and is called the gradient of f at point (x_0, y_0).

An important interpretation of the gradient.

Look at

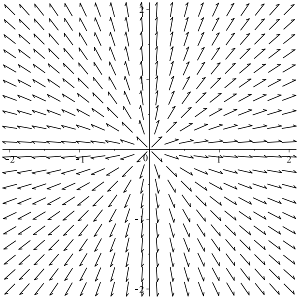
$$D_{\mathbf{u}}f(x_0, y_0) = |\nabla f(x_0, y_0)| \cos(\theta)$$

Again the question: In which direction **u** is $D_{\mathbf{u}}f(x_0, y_0)$ a maximum?
answer: "the gradient direction"
Some questions:

1. In what direction do you have to take your (unit) step for minimizing $D_{\mathbf{u}}f(x_0, y_0)$?
2. and in what direction do you have to take your (unit) step(s) such that value $f(x_0, y_0)$ is "unaffected"?
3. Make a walk in the xy-plane which minimize in every step the value of $\sqrt{x^2 + y^2}$!

A gradient-plot.

In next picture you see a Maple constructed gradient plot for the function $f(x,y) = \sqrt{x^2+y^2}$:



If you follow the "opposite gradient vectors" starting at a random point you "always" end at $(0,0)$ which belongs to the minimum value of f.

Generalizations:

- In $\mathbf{R}^n$ (n the input variables $x_1, x_2 ... x_n$ are involved) the directional derivative is calculated as:

$$D_{\mathbf{u}} f(\mathbf{p}) = \nabla f(\mathbf{p}) \cdot \mathbf{u}$$

- with $\mathbf{u} = \langle u_1, \cdots u_n \rangle$ a vector in $\mathbb{R}^n$,
- with $\mathbf{p} = \langle p_1, \cdots p_n \rangle$ a vector in $\mathbb{R}^n$,
- with $\nabla f(\mathbf{p}) = \langle f_{x_1}(\mathbf{p}), f_{x_2}(\mathbf{p}), \cdots f_{x_n}(\mathbf{p}) \rangle$,
- with dot product $\mathbf{a} \cdot \mathbf{b}$ defined as $a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$,
- Interpretation for gradient $\nabla f(\mathbf{p})$ is: If we choose $\mathbf{u}$ as $\frac{\nabla f(\mathbf{p})}{|\nabla f(\mathbf{p})|}$ then for given $\mathbf{p}$ is $D_{\mathbf{u}} f(\mathbf{p})$ a maximum
- Remark: The the gradient vector lives in the input space (domain).

An Example.

Question

Given a rectangular box with sizes 8[m] for the length l, 4[m] for width w and 2[m] for height h.

We change l, w and h such that $\triangle l$, $\triangle w$, $\triangle h$ fulfill the condition:

$$\sqrt{(\triangle l)^2 + (\triangle w)^2 + (\triangle h)^2} = 1.$$

Estimate which $\langle \triangle l, \triangle w, \triangle h \rangle$ gives largest volume?

An Example.

Answer:

- We consider the volume V as function:$V(l, w, h) = l \cdot w \cdot h$.
- We are "sitting" in $P(8, 4, 2)$.
- $\nabla V = \langle wh, lh, lw \rangle$ so in P this equals to $\langle 8, 16, 32 \rangle$
- So $D_{\mathbf{u}}V(8, 4, 2)$ is a maximal for $\mathbf{u} = \langle \frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{4\rangle}{\sqrt{21}} \rangle$
  A good estimate is $\triangle l = \frac{1}{\sqrt{21}}, \triangle w = \frac{2}{\sqrt{21}}, \triangle l = \frac{4}{\sqrt{21}}$.

Approximating extreme values with the gradient

Given some function $f(x_1, \cdots x_n)$.

1. Look for a suitable starting point $(q_1, q_2, \cdots q_n)$ (in the domain)
2. Choose a suitable stepsize S (some positive number)
3. Define recursion relation (vector form)

$$\mathbf{X}_n = \mathbf{X}_{n-1} \pm S \nabla f(\mathbf{X}_{n-1})$$

with starting value $\mathbf{X}_0 = \langle q_1, q_2, \cdots q_n \rangle$. Herein we choose $+$ if we are looking for a maximum, we choose $-$ if we are looking for a minimum.

For application see Maple Demo