Research Seminar 2.
Zero-Sum Games as LP

- Mathematical Formulation of LP
- Defining Objectives for Column and Row
- Duality
- LP Solutions for Zero-Sum Games

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Sample Zero-Sum Game

Colin

Rose	A	B	C
A	-4	-9	8
B	0	4	5
C	-10	-8	-2
D	3	-7	-3
E	-9	-3	5

Colin

Rose	A	B	C
A	-4	-9	8
B	0	4	5
C	-10	-8	-2
D	3	-7	-3
E	-9	-3	5

Mixed Strategy Solution to the Game

		Colin A	Colin B	Row Differences	Rose Oddments	Rose Probabilities
Rose	B	0	4	0 -4 = -4	10	5/7
	D	3	-7	3 - -7 = 10	4	2/7

- Eliminate dominated strategies
- No saddle point
- William’s oddments offers a quick solution

Definition of a Linear Program (LP)

Minimize	cx	Objective Function
with respect to	x	Decision Variables
Subject to	$Ax = b$	Constraints
	$x >= 0$	Parameters A, b, c

Solutions to the linear program can be automatically found by computer

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Rose's Objectives

- Rose seeks to maximize the value of the game

$$
\begin{array}{lllll}
\text{maximize} & \nu & & & \\
\text{with respect to} & B, D, \nu & & & \\
\text{subject to} & & & 4D & \geq \nu \\
& 3B & - & 7D & \geq \nu \\
& B & + & D & = 1 \\
& & & B, D & \geq 0
\end{array}
$$

Algebraic form

$$
\begin{array}{ll}
\text{maximize} & Z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B \\ D \\ \nu \end{bmatrix} \\
\text{with respect to} & B, D, \nu \\
\text{subject to} & \begin{bmatrix} 0 & 4 & -1 \\ 3 & -7 & -1 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} B \\ D \\ \nu \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \\
\text{and} & \begin{bmatrix} B \\ D \\ \nu \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{array}
$$

Matrix form

Duality

- Every linear program has a dual
- Every maximization problem can be recast as a minimization problem with an appropriate change in variables
- Duality plays an important role in sensitivity analysis of linear programming (LP)
- Duality is also related to the minimax theorem of game theory

Primal and Dual Problems

maximize $Z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B \\ D \\ \nu \end{bmatrix}$

with respect to B, D, ν

subject to $\begin{bmatrix} 0 & 4 & -1 \\ 3 & -7 & -1 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} B \\ D \\ \nu \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$

and $\begin{bmatrix} B \\ D \\ \nu \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Primal form

minimize $W = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$

with respect to y_1, y_2, y_3, y_4

subject to $\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 0 & 4 & -1 \\ 3 & -7 & -1 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \geq \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

and $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Dual form

Colin's Objectives

- Colin seeks to keep down the "slack" variables y
- The slack variables are the excess payments claimed by Rose

Minimize the slack
$$W = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

subject to

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 4 & -1 \\ 3 & -2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \leqslant \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \geqslant \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

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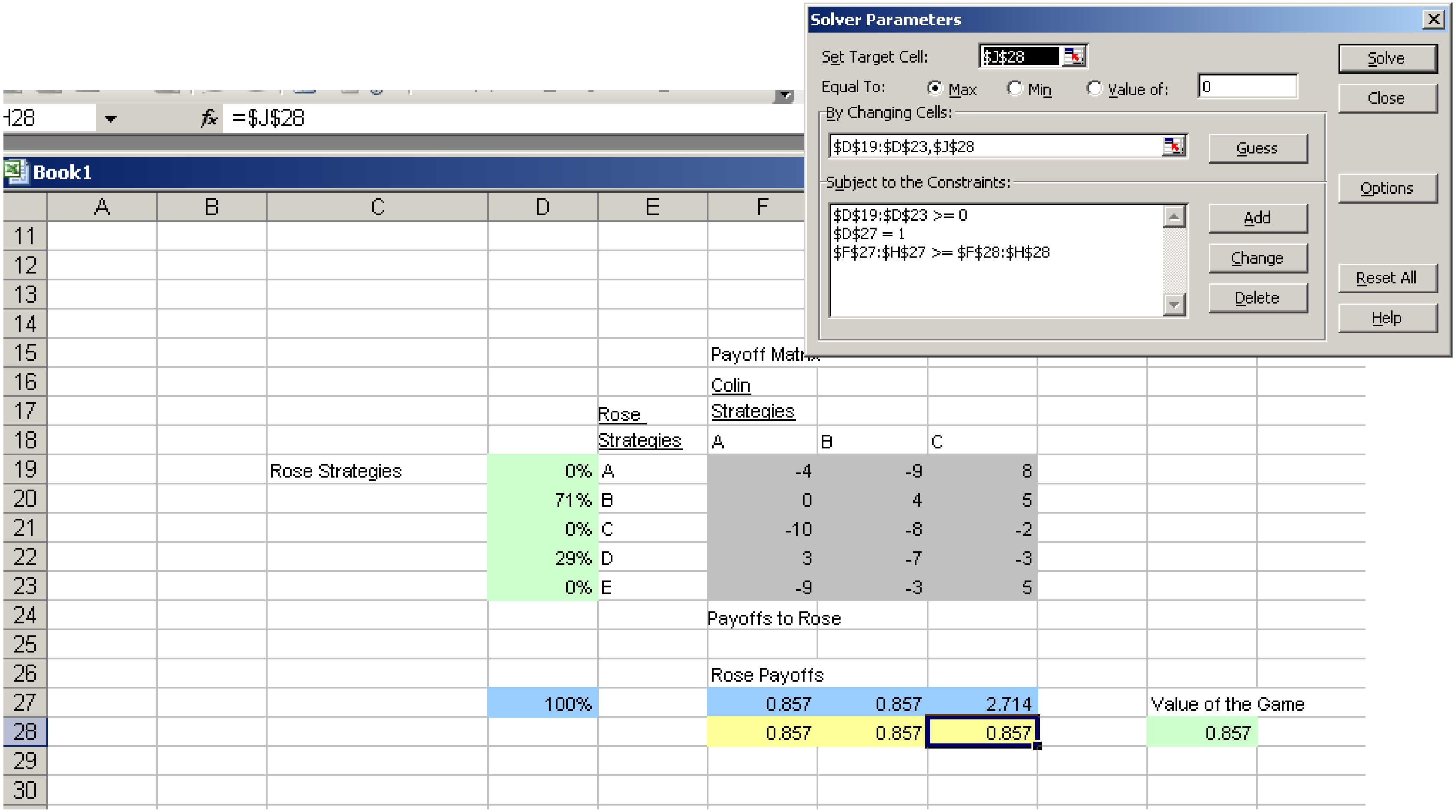
Zero-Sum Games and Linear Program

- Colin and Rose have two linear programs
- One is the primal problem and one is the dual
- Thus, all zero-sum games can be reduced into LP form
- Advantages of thinking in terms of LP
  - Computer implementation
  - Interpretation of solution concepts
  - Modeling flexibility
  - Existing LP problematiques

Zero-Sum Implementation in Excel



Screenshots by Scott Cunningham, using Microsoft Excel Solver.

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Weakly Dominated and Strongly Dominated

- Strongly dominated solutions occur when one strategy profile is strictly better than another
- Weakly dominated solutions occur when one strategy profile is greater than or equal to another

Colin

Rose	A	B	C
A	−4	9	8
B	0	4	5
C	−10	−8	−2
D	3	−7	−3
E¿	−9	−3	5

A dominates B

B dominates C

B dominates E

B dominates C

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Weak Domination and Adjacent LP Solutions

- Rose B weakly dominates Rose E
- Weak domination occurs through column C
- Note that we later throw out Colin's C strategy
- Weak domination can lead to alternative equilibria and adjacent LP solutions

Colin

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A	−4	9	8
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A dominates B

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B dominates C

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