A short introduction to modal logic for games

EPA 2142

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December 2009

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TBM, B4.280

Introduction

Formal representations of games:

- Game matrix

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- Game tree

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Can serve as models to talk about in a formal logical language.

Logic $\equiv$ formal language $+$ semantics $+$ syntactic reasoning rules

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In a formal language:

Truth judgement (*proposition*): P, $\neg P$, $P \& Q$, $P \rightarrow Q \ldots$
'Possibly P': $\Diamond P$
'Necessarily P': $\Box P$

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A Kripke model $M = (W, R, \pi)$ consists of:

- a set of worlds (W)
- a accessibility relation between those worlds ($R \subseteq W \times W$)
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How about: $\Box P \rightarrow P$?

Flavours of modal logic

Modal logic comes in different flavours (e.g. by restricting to certain types of accessibility relations)

- knowledge: epistemic logic (with one relation R_i for each agent i: which worlds does i consider possible?)
- beliefs: doxastic logic,
- obligations: deontic logic,
- actions: dynamic logic,
- time: temporal logic (linear or branching time),
- ability: strategic logic...

and combinations of the above (e.g. for multi-agent systems: temporal epistemic logic).

Example: epistemic logic

Epistemic logic for multiple agents, $i \in \{1, \ldots, n\}$:

$$\text{interpret } \Box_i P \text{ as "agent } i \text{ knows } P\text{"}$$

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$M, w \models K_i\phi$ iff ϕ holds in all worlds that i can't distinguish from w

(Example: PD-matrix, imagine C and R interrogated at the same time in separate rooms.)

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[Philosophical problem: if ϕ is universally true, then $K_i \phi$ is universally true. Cf. existence of a winning strategy in chess.]

Extensions and variations on basic modal logics

Some examples, most relevant for games:

- In multi-agent settings: group knowledge (Common Knowledge, Distributed Knowledge)
- Dynamic logics: abstract programs as modalities ($[\pi]\phi$: whenever π terminates, it is in a state satisfying ϕ)
- Game Logic: abstract games as modalities ($\langle\gamma\rangle\phi$: 'Angel has a ϕ-strategy in game γ')
- Coalition Logic

  ...

Game theory meets modal logic

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Modal logic and its extensions give 'new sights of a familiar landscape':

- game trees as Kripke models,
- (dynamic) epistemic logic to analyze imperfect information games
- Game Logic: reasoning about strategies
- Coalition logic: reasoning about coalitional power

Coalition Logic (Marc Pauly, 2001)

Elements:

- Game frames: transition systems where the transitions are determined by a strategic game form in each state (and where the outcomes are, again, states)
- Modal logic can be used to reason about how the system possibly or necessarily will evolve
- Coalition logic: state what outcomes coalitions can force coalitional power

Main construct, for a subset of agents C:

$$\langle \mathbf{C} \rangle \phi\text{: coalition } \mathbf{C} \text{ can force } \phi \text{ to be true.}$$

Expressions in Coalition Logic

Example

C and R must choose between outcomes P and Q.
We want a choice-mechanism satisfying:

- Both P and Q should be possible outcomes of the collective choice
- but not both simultaneously
- neither C nor R should be able to dictate the choice

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$\neg\langle\{C,R\}\rangle P\&Q$

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$\neg\langle\{C, R\}\rangle P \& Q$
$\neg\langle\{C\}\rangle P$; $\neg\langle\{C\}\rangle Q$; $\neg\langle\{R\}\rangle P$; $\neg\langle\{R\}\rangle Q$

(This slide is an adaptation the ESSLLI-slides of Agotnes & Jamroga.)

Semantics: effectivity functions

Coalitional power: Formalised by effectivity functions, mapping each coalition to the set of outcomes it can reach.

$X \in E(\mathbf{C})$: coalition $\mathbf{C}$ can force the outcome to be in $X \subseteq S$

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Game form induces effectivity function.
Prisoner's Dilemma, cf. p.74 of Straffin
PD induces the effectivity function E_{PD}:

$$
\begin{array}{rcl}
E_{PD}(\emptyset) & = & \{S\} \\
E_{PD}(Colin) & = & \{\{(R,R),(T,S)\},\{(S,T),(U,U)\}\}^{+} \\
E_{PD}(Rose) & = & \{\{(R,R),(S,T)\},\{(T,S),(U,U)\}\}^{+} \\
E_P D(\{Colin, Rose\}) & = & \mathcal{P}(S) - \{\emptyset\}
\end{array}
$$

where X^{+} is the closure of X under supersets.

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Recommended reading

- Fagin at al., *Reasoning About Knowledge*, 1994.
- van der Hoek & Pauly: *Modal Logic for Games and Information*, Handbook of Modal Logic, 2006.
- Thomas Agotnes and Wojtek Jamroga: Slides of ESSLLI-course *Modal Logics for Games and Multi-Agent Systems* (`http://www2.in.tu-clausthal.de/~wjamroga/courses/GamesMAS2008ESSLLI/`)
- Johan van Benthem, *Logic in Games*, to appear (2010?).