Lineaire Schakelingen

ET1300

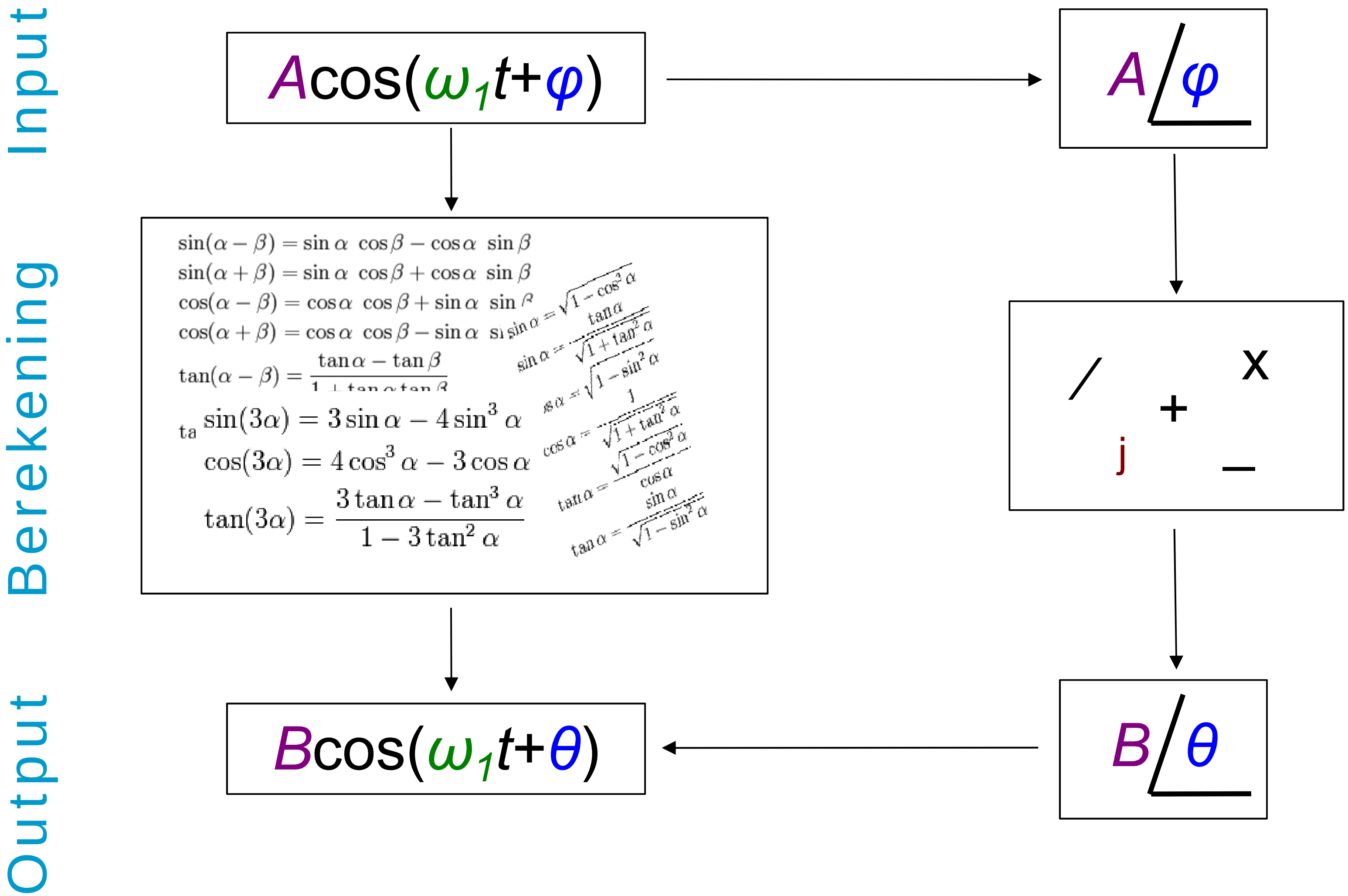
Instructie, week 2.4

Slides by Marijn van dongen, et al.

Vandaag

- **Laplace transformatie**
- **Inverse Laplace transformatie**
- Laplace voor circuit-analyse (volgende keer)

Herinnert u zich deze nog?



Tot nu toe

- DC bronnen (Steady state)
- AC bronnen (Steady state)
- Eerste- en tweede orde transiente circuits (Schakelverschijnselen)

Maar het kan altijd moeilijker!

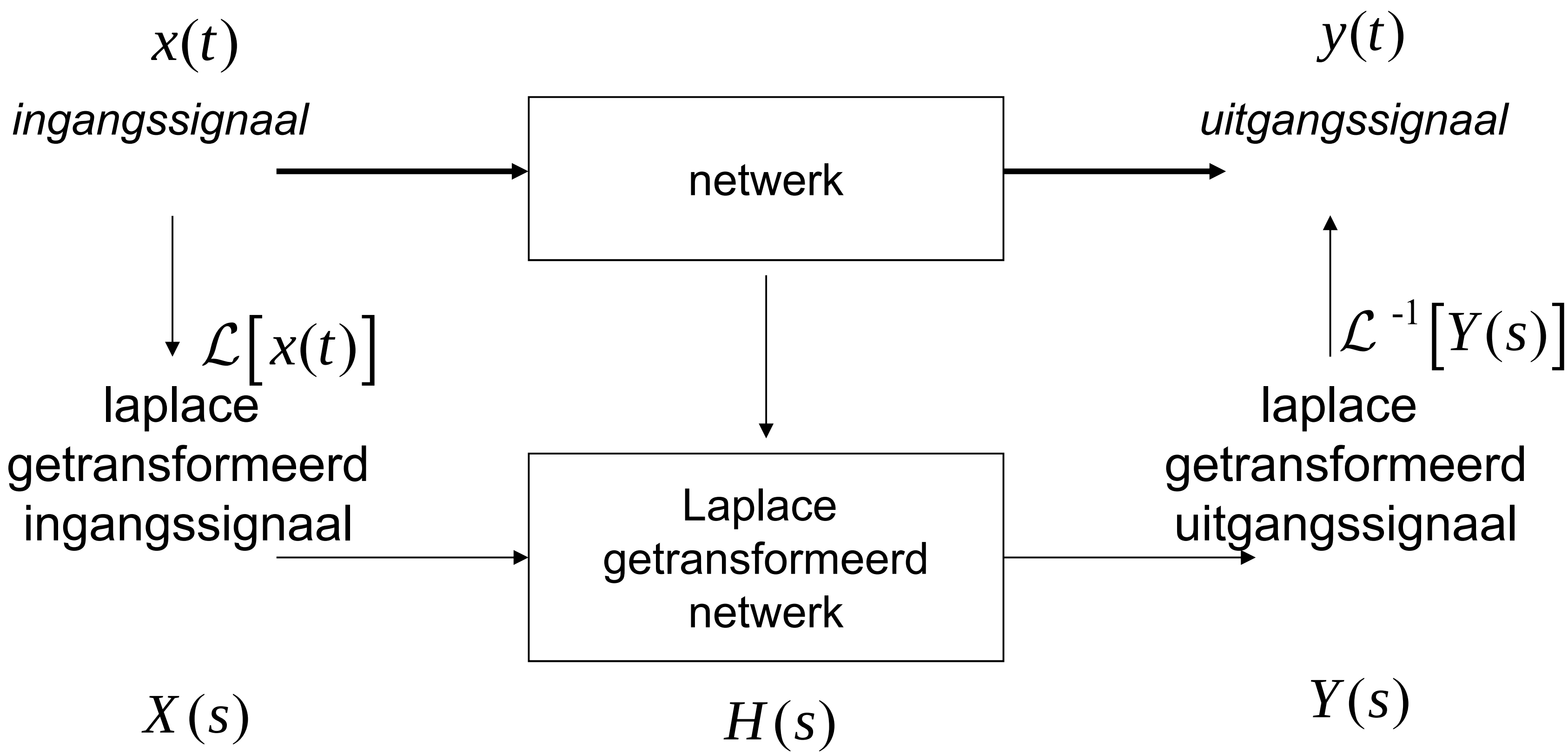
- Schakel- en impuls-bronnen
- Exponentiele bronnen
- etc...

Laplace transformatie

Een niet zo goede definitie:

- Is een 'uitbreiding' op het frequentiedomein
- Is een wiskundig trucje om systemen met ingewikkelde signalen door te rekenen

Het 'Idee'



x(t)
ingangssignaal
netwerk
y(t)
uitgangssignaal
ℒ[x(t)]
laplace getransformeerd ingangssignaal
Laplace getransformeerd netwerk
ℒ -1 [Y(s)]
laplace getransformeerd uitgangssignaal
X(s)
H(s)
Y(s)

Laplace transformatie

Een betere definitie:

- Voor lineaire systemen kun je een tijddomein functie naar het laplace domein transformeren middels:

$$L\left[f(t)\right] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

- En terugtransformeren middels:

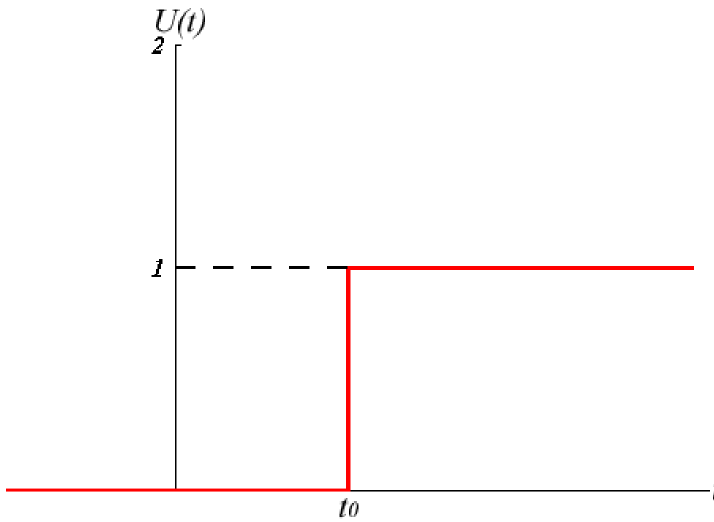
$$L^{-1}\left[F(s)\right] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\omega}^{\sigma_1 + j\omega} F(s)e^{st}ds$$

Vele functie relaties zijn al bepaald:

Stapfunctie: $u(t-t_0)$ of $\varepsilon(t-t_0)$

$u=0$ voor $t<t_0$

$u=1$ voor $t \geq t_0$



Impulsfunctie: $\delta(t-t_0)$

$\delta=0$ voor $t \neq t_0$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

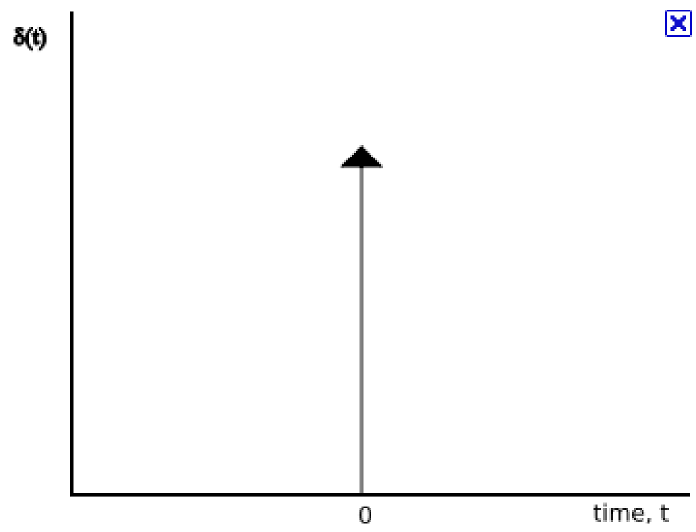


TABLE 15.1

Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(x)\,dx$	$\frac{1}{s} F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds} F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t + nT)$	$\frac{F_1(s)}{1 - e^{-sT}}$
Initial value	$f(0)$	$\lim_{s \to \infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s \to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

TABLE 15.2

Laplace transform pairs.*

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

Example

Obtain the Laplace transform of $f(t) = \delta(t) + 2u(t) - 3e^{-2t}u(t)$.

Solution:

By the linearity property,

$$F(s) = \mathcal{L}[\delta(t)] + 2\mathcal{L}[u(t)] - 3\mathcal{L}[e^{-2t}u(t)]$$

$$= 1 + 2\frac{1}{s} - 3\frac{1}{s+2} = \frac{s^2 + s + 4}{s(s+2)}$$

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$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
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The inverse Laplace transform

Suppose $F(s)$ has the general form of

$$F(s) = \frac{N(s)}{D(s)}$$

Steps to Find the Inverse Laplace Transform:

1. Decompose $F(s)$ into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Table 15.2.

Single poles

$$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \longrightarrow F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n}$$

General solution

$$f(t) = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t})u(t)$$

Repeated poles

$$F(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} + \frac{k_1}{s+p} + F_1(s)$$

General solution

$$\longrightarrow f(t) = \left(k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{2!}t^2 e^{-pt} + \cdots + \frac{k_n}{(n-1)!}t^{n-1}e^{-pt}\right)u(t) + f_1(t)$$

Complex poles

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s) \longrightarrow f(t) = (A_1 e^{-\alpha t}\cos\beta t + B_1 e^{-\alpha t}\sin\beta t)u(t) + f_1(t)$$

General solution

Example inverse Laplace transform

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

Solution:

The inverse transform is given by

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left(\frac{3}{s}\right) - \mathcal{L}^{-1}\left(\frac{5}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{6}{s^2+4}\right)$$

$$= (3 - 5e^{-t} + 3\sin 2t)u(t), \qquad t \geq 0$$

where Table 15.2 has been consulted for the inverse of each term.

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