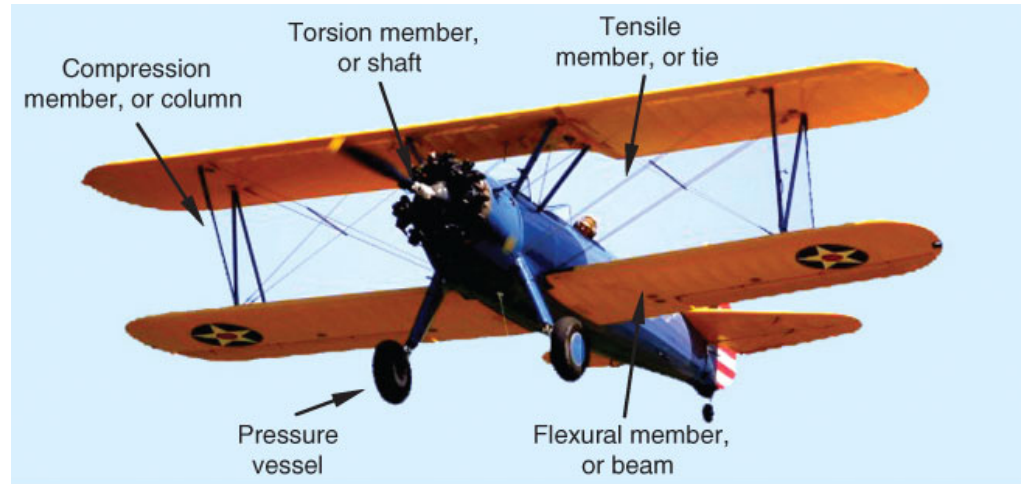


Chapter 4

Stiffness and weight: density and elastic moduli



4.1 Introduction and synopsis

- Stress causes strain (shape change).
- Modes of loading:
 - ties carry *tension* (cables)
 - columns carry *compression* (tubes are less prone to buckling (**knikken**) than solid rods)
 - beams carry *bending* moments (wing spar)
 - shafts carry *torsion*, pressure vessels contain *internal pressure*.
- Stiffness: resistance to shape change that is elastic (returns to original shape when load is released (measured by elastic modulus, E)).

- Strength is resistance to permanent deformation or total failure. Measured by elastic limit, σ_y , or tensile strength, σ_{ts} .
- Often stiffness and strength are combined with density, ρ .
- The first material property chart considered: the modulus-density chart.
- Note: modulus and density for pure materials cannot be tailored; a function of the atomic mass, atomic packing, and bond strength.

4.2 Density, stress, strain, and moduli

- *Density* – mass/volume.
- Accurate measurements of density can be made by the double weighing method. When immersed the sample feels an upward force equal to the weight of the liquid it displaces.

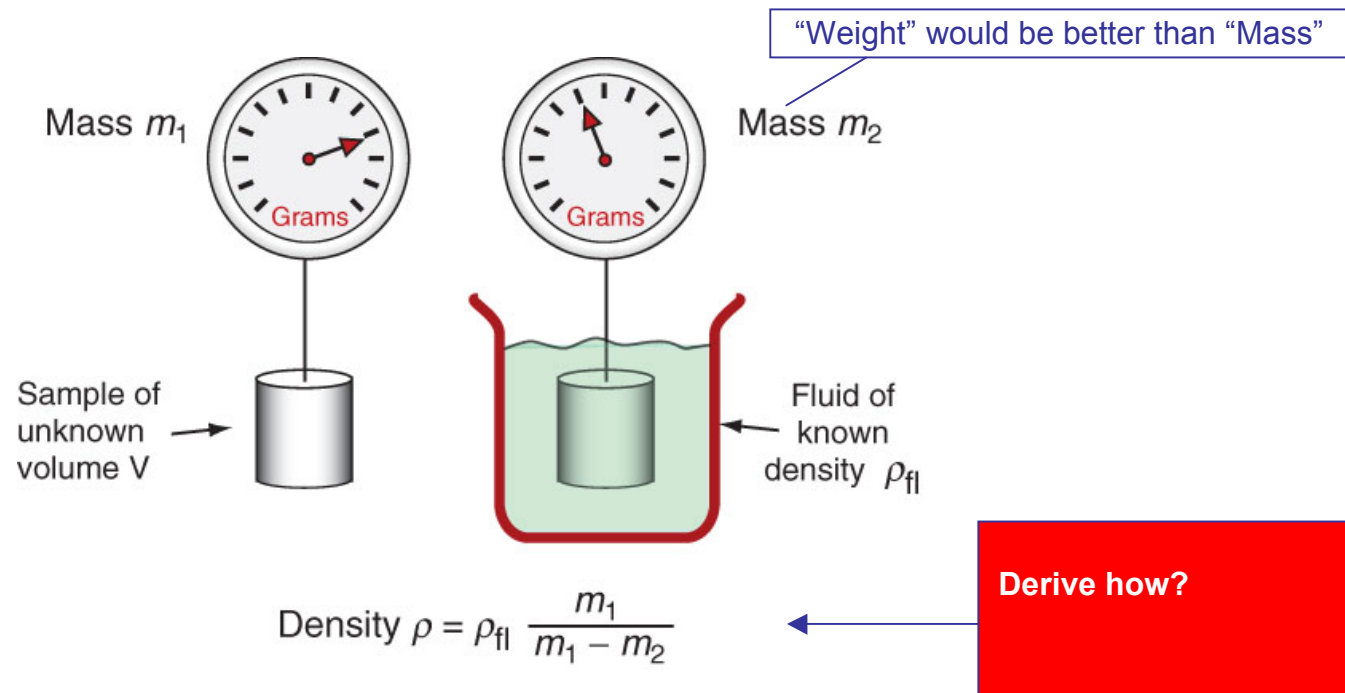


Figure 4.1 Measuring density by Archimedes' method.

4.2 Density, stress, strain, and moduli

- *Density* – mass/volume.
- Accurate measurements of density can be made by the double weighing method. When immersed the sample feels an upward force equal to the weight of the liquid it displaces.

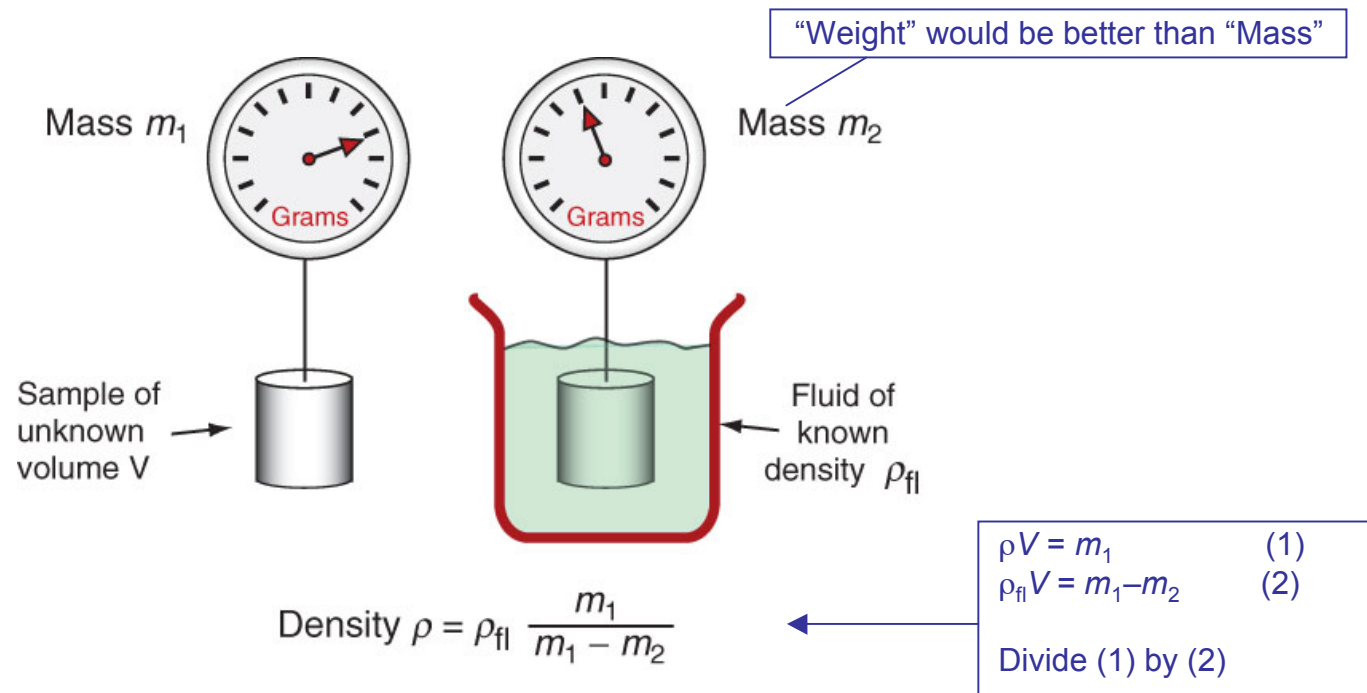
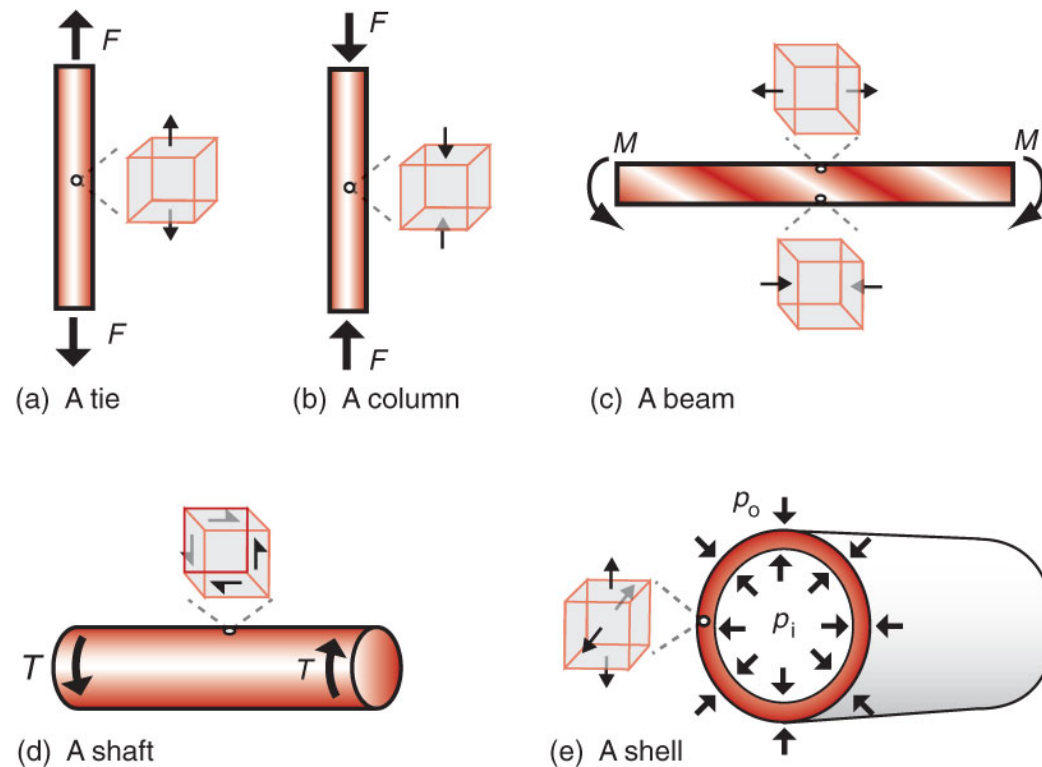


Figure 4.1 Measuring density by Archimedes' method.

Figure 4.2 Modes of loading and states of stress.



- *Modes of loading*
tie: axial tension
column: axial compression
beam bending: axial tension on one side, compression on the other
shafts: twisting or torsion (shear)
pressure difference across a shell: bi-axial tension or compression.

Stress: (*spanning*) force per area, F/A .

- Simple **compression** and **tension**: applied normal to a face of a reference element, $\sigma = F/A$.
- Forces applied parallel to a face create a state of **shear** in the element with shear stress, $\tau = F_s/A$.
- Equal tensile or compressive stress on all six faces: **hydrostatic pressure**, p .
- Units are MPa (MN/m^2) = 10 bar

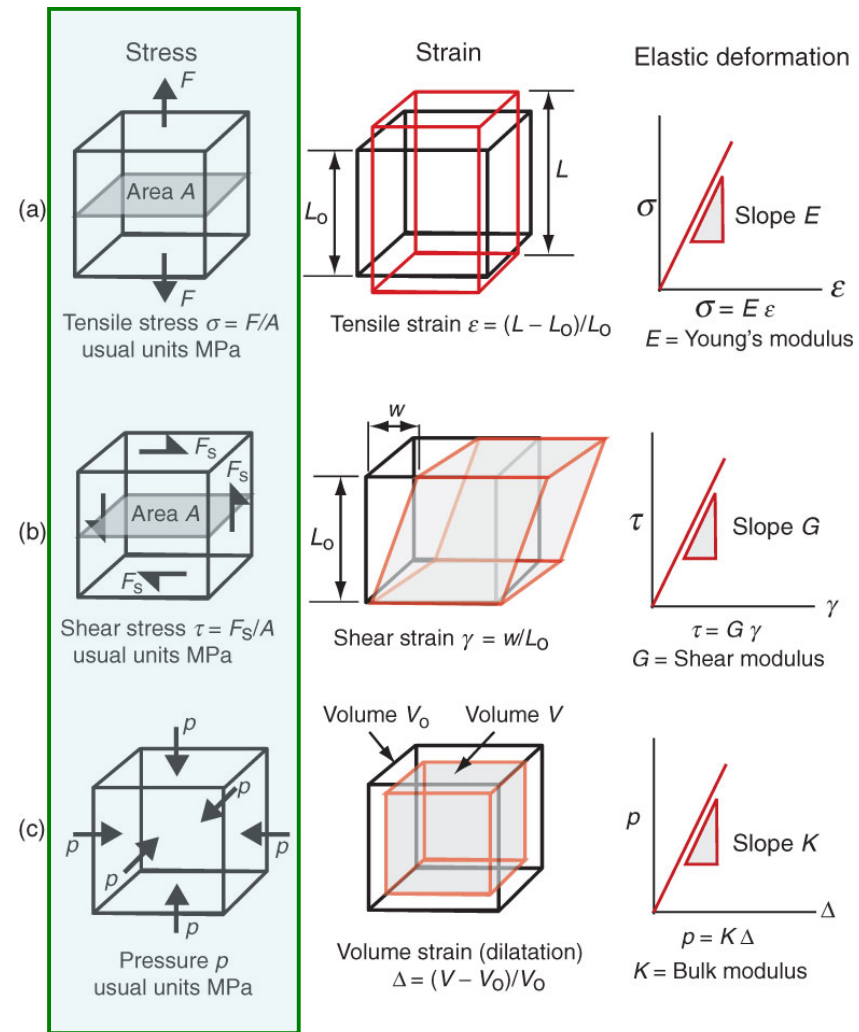


Figure 4.3 The definitions of stress, strain and elastic moduli.

Sign convention: **P**ressure = **p**ositive when **p**ushing
Stress has other **s**ign

Example

Example 4.1

A brick chimney is 50 m tall. The bricks have a density of $\rho = 1800 \text{ kg/m}^3$. What is the axial compressive stress at its base? Does the shape of the cross-section matter?

Strain (*rek*) is a response to stress.

- It is dimensionless.
- Tensile or compressive strain:

$$\varepsilon = \delta L / L_0$$
- Shear strain:

$$\tan(\gamma) = w / L_0 \approx \gamma$$
- Volumetric strain (dilatation):

$$\Delta = \delta V / V$$

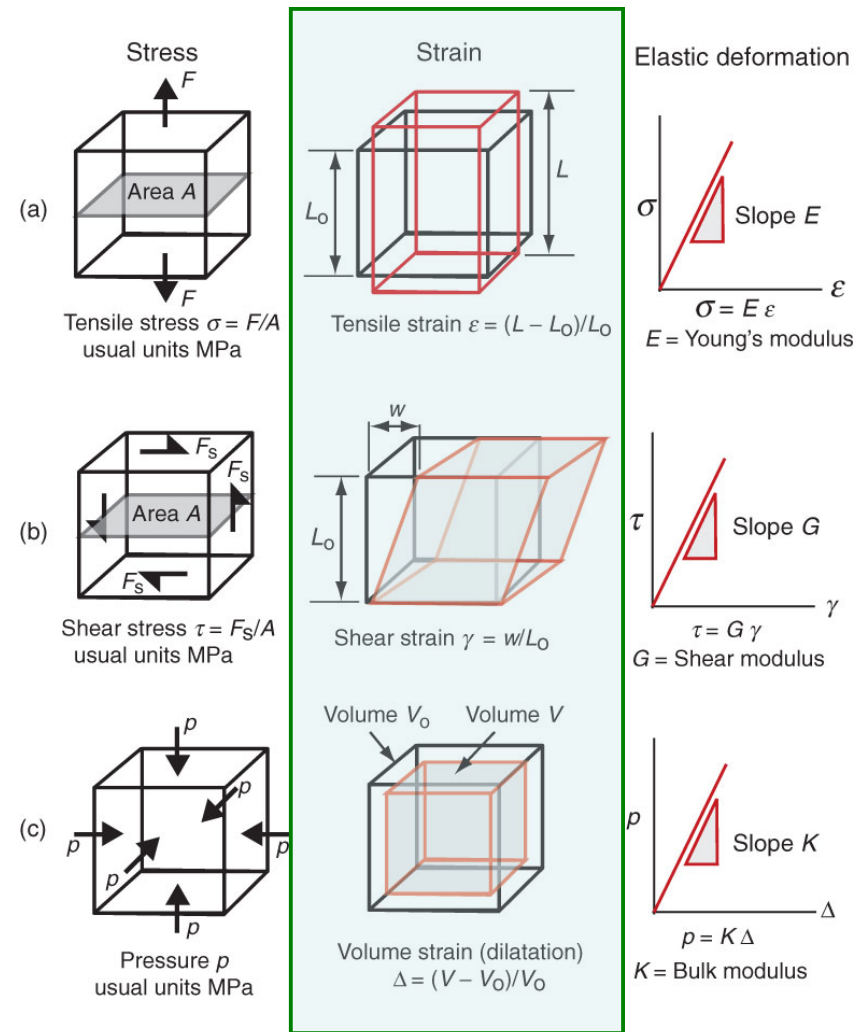


Figure 4.3 The definitions of stress, strain and elastic moduli.

Moduli

- In the linear elastic regime strain is proportional to stress.
- Tensile or compressive: $\sigma = E\varepsilon$, where E is Young's modulus. (Eq. 4.6)
- Shear: $\tau = G\gamma$, where G is the shear modulus. (Eq. 4.7)
- Pressure: $p = K\Delta$, where K is the bulk modulus. (Eq. 4.8)

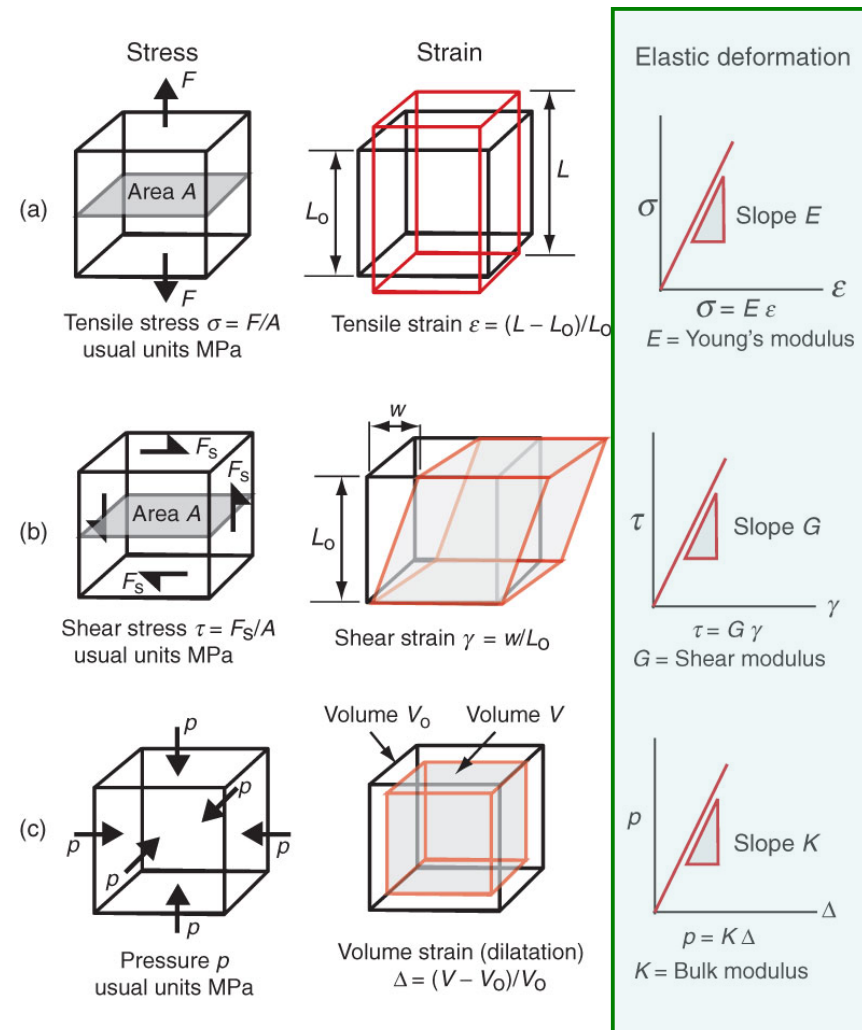


Figure 4.3 The definitions of stress, strain and elastic moduli.

- Representative stress-strain curves for different classes of materials.
- Note the elastic limit and yield and tensile strengths, and magnitude of the strains.
- A_0 is the *initial* cross sectional area
- A is the *actual* cross sectional area, which may become smaller than A_0 as the loading proceeds

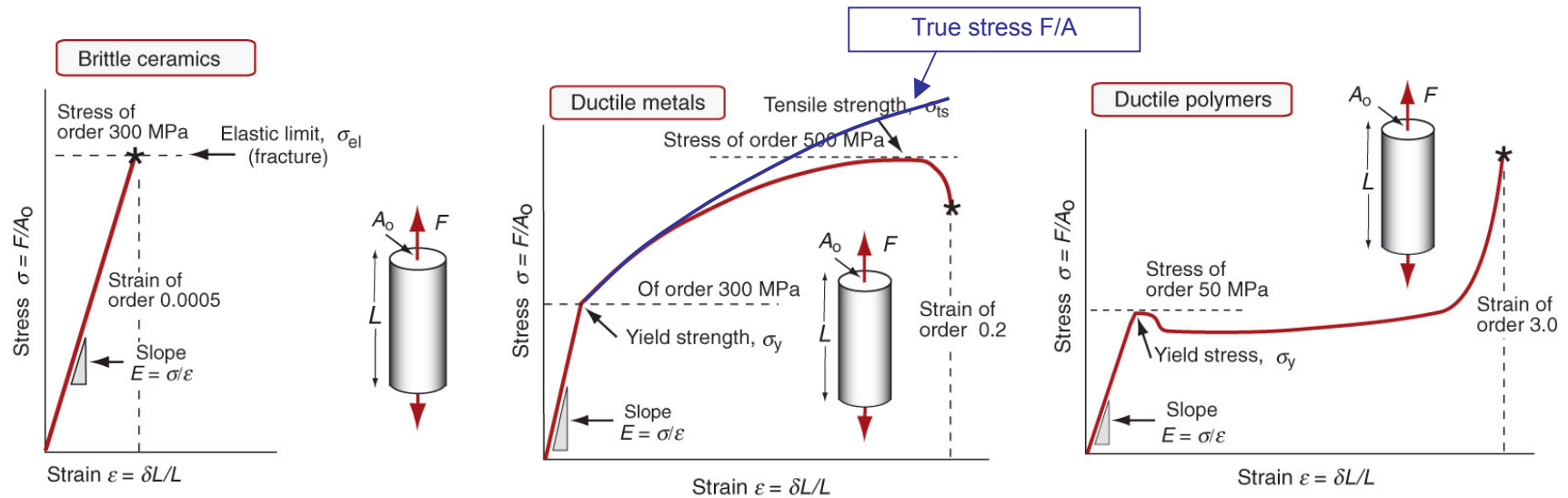


Figure 4.4 Tensile stress–strain curves for ceramics, metals and polymers.

Relating the moduli through Poisson's ratio, ν .

- Generally stretching in one direction leads to contraction in the other two.

ν is the negative of the ratio of the lateral or transverse strain, ϵ_t , to the axial strain, ϵ .

In tensile loading $\nu = -\epsilon_t/\epsilon$. It is usually about 1/3.

- For an isotropic material $G = \frac{E}{2(1 + \nu)}$ $K = \frac{E}{3(1 - 2\nu)}$
- With $\nu = 1/3$: $G \sim 3E/8$ and $K \sim E$
- For elastomers $\nu \sim 1/2$ and $G \sim E/3$ and $K > E$; rubber is easy to stretch in tension (low E) but if the shape is constrained or the load is hydrostatic it is very stiff (high K).

Example

Example 4.2

The chimney in Example 4.1 has bricks with Young's modulus $E = 25 \text{ GPa}$, and Poisson's ratio $\nu = 0.2$. What are the axial and transverse strains at the bottom of the chimney?

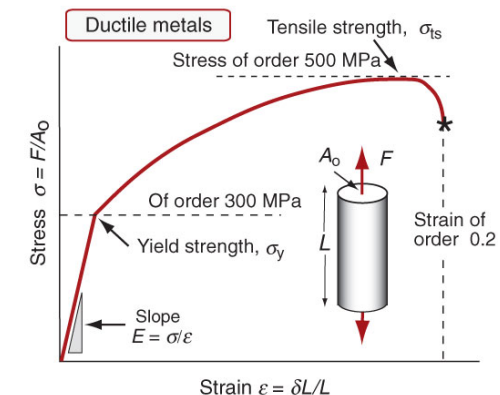
Elastic energy

- A force F acting through a displacement dL does work FdL .
- A stress $\sigma = F/A_0$ acting through a strain increment $d\varepsilon = dL/L$ does work per unit volume $dW = FdL/A_0L = \sigma d\varepsilon$ (J/m²). In an elastic material this work is stored as elastic energy.
- The work done per unit volume as the stress is increased from zero to a final value σ^* is just the area under the stress-strain curve.

Whole stress-strain curve

Elastic part of the stress-strain curve only

$$W = \int_0 \rightarrow \sigma^* \sigma d\varepsilon = \int_0 \rightarrow \sigma^* (\sigma/E) d\sigma = \sigma^{*2}/2E$$



- *Measuring E* : usually done dynamically, vibrating beam or velocity of sound in the material; both depend on $(E/\rho)^{1/2}$.

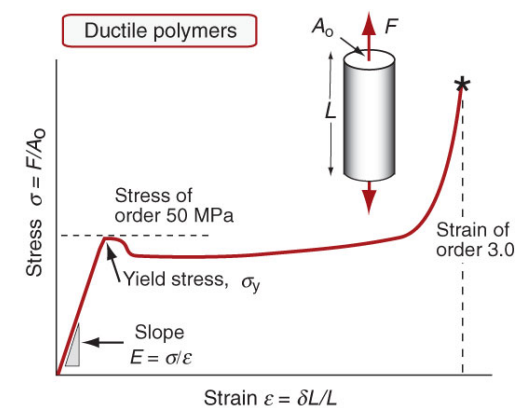
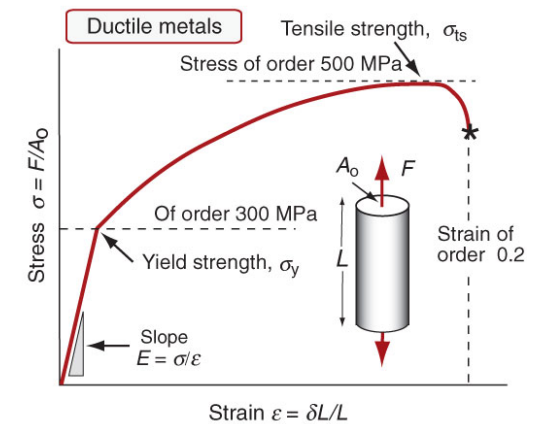
PI question

Why is the expression

$$W = \int_{0 \rightarrow \sigma^*} (\sigma/E) d\sigma$$

for the work per unit volume W needed to strain a certain material from zero stress to a final stress $\sigma^* = F^*/A_0$ only valid for σ^* not larger than the yield stress σ_y ?

1. For larger strains, the integral should be taken along the ϵ axis
2. Because, if you remove the force, you always get some of the energy back
3. The expression is valid until the stress-strain curve curves downward
4. For larger strains, σ and ϵ are no longer proportional

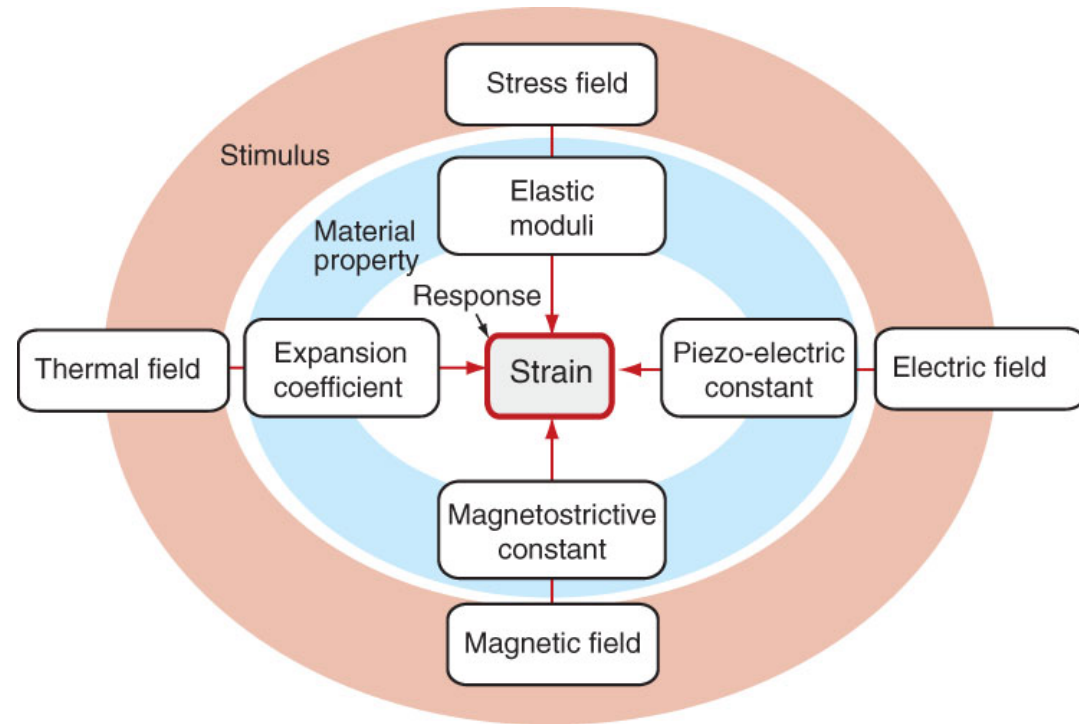


Example

Example 4.3

A steel rod has length $L_o = 10$ m and a diameter $d = 10$ mm. The steel has Young's modulus $E = 200$ GPa, elastic limit (the highest stress at which it is still elastic and has not yielded) $\sigma_e = 500$ MPa and density $\rho = 7800$ kg/m³. Calculate the force in the rod, its extension and the elastic energy per unit volume it stores when it is stretched so that the stress in it just reaches the elastic limit. Compare the elastic energy per unit mass with the chemical energy stored in gasoline, which has a heat of combustion (calorific value) of 43 000 kJ/kg.

Figure 4.5 Stimuli leading to strain.



Stress-free strain

- Other stimuli cause strain (e.g., magnetic and electric fields: magnetostrictive and piezo-electric materials). These strains are generally very small but can be accurately controlled and at high frequency; used in precise positioning and sensors.
- Thermal expansion: thermal strain is proportional to the temperature change, $\epsilon_T = \alpha \Delta T$. α is the thermal expansion coefficient.
- ‘Stress-free’ strains (piezo, thermal, etc) do cause stresses: if the material is constrained. Misleading term.

4.3 The big picture: material property charts

- *The modulus-density chart*; the modulus spans 7 decades while densities vary over a factor 2000.

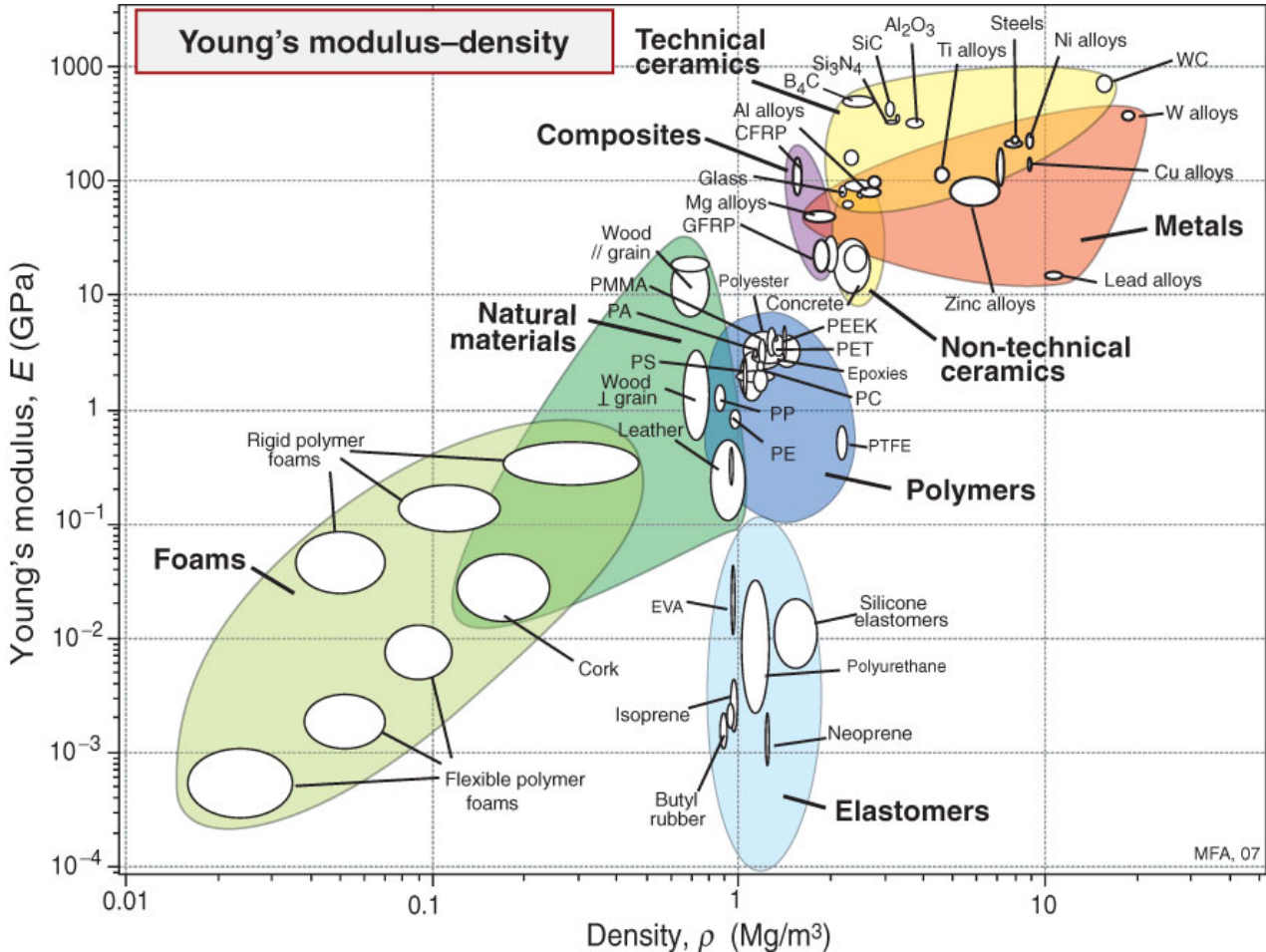
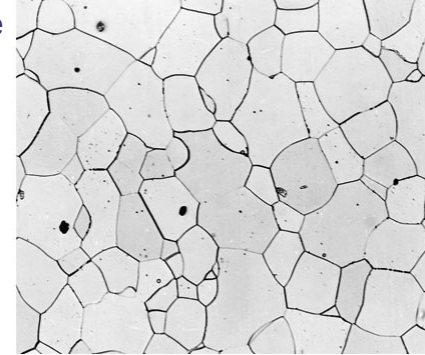


Figure 4.6 The modulus–density chart.

Grains in a metal, $\sim\mu\text{m}$ diameter scale

Anisotropy



- Glasses and many polymers are generally disordered down to the atomic scale, with little directionality of properties.
- Polycrystalline metals and ceramics have more or less randomly oriented 'grains' (= small crystals) and are also roughly isotropic on average.
- For single crystals and drawn polymers and fibers anisotropy becomes important – properties depend strongly on direction of measurement.
- Property charts reflect anisotropy by having property bubbles for each loading direction (e.g. *Wood*, previous chart).

4.4 The science: what determines the density and stiffness?

- *Density*; atoms differ greatly in mass but not so much in size – therefore density is largely controlled by atomic weight, with secondary effects from size and packing.
- Metals: high density (heavy atoms, Al(M=27)→Ti(48)→Fe(56)→Ag(108)→Pb(207), tightly packed)
- Polymers: low density (carbon (12) and hydrogen(1)-based, loosely packed)
- Ceramics: intermediate (contain high fraction of light elements, e.g. C(12), N(14), O(16), medium packing)
- Atom diameters in nm: very small (0.05, H), small (0.15, C), medium (0.25, many metals), large (just a few: 0.4, K, Ba)

*Atom packing in **metals** and the unit cell*

- Crystalline structures: view as arrangements of hard, spherical balls.
- Close-packed structures: a common feature in 2D and 3D. Stacking sequence determines HCP (hexagonal close-packed, e.g., Mg, Ti) versus FCC (face-centered cubic, e.g., Cu, Ag, Au, Al). Both with 0.74 packing fraction.

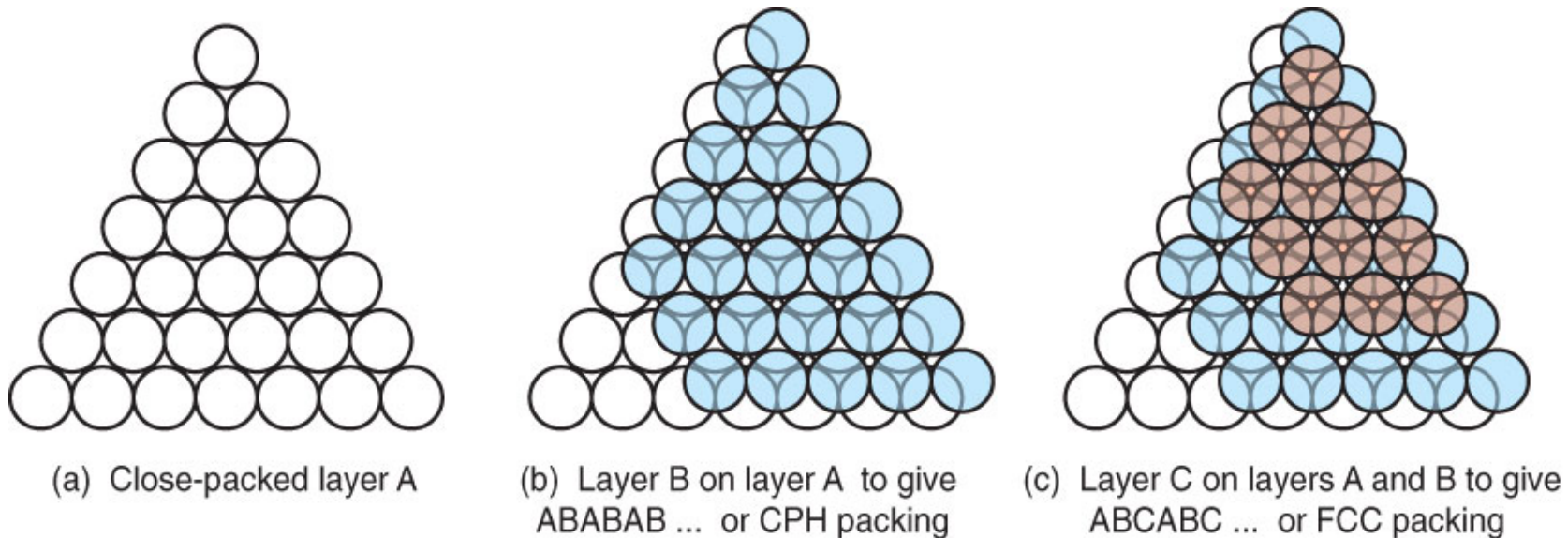


Figure 4.8 (a) Close-packed layer of spheres, layer A. (b) Second layer, B, nesting in the first; repeating this sequence gives ABAB . . . or CPH stacking. (c) Third layer, C, can be nested so that it does not lie above A or B; if repeated this gives ABCABC: FCC stacking.

Another common arrangement

- BCC – body-centered cubic; not close-packed (0.68). Fe, Ta, W.

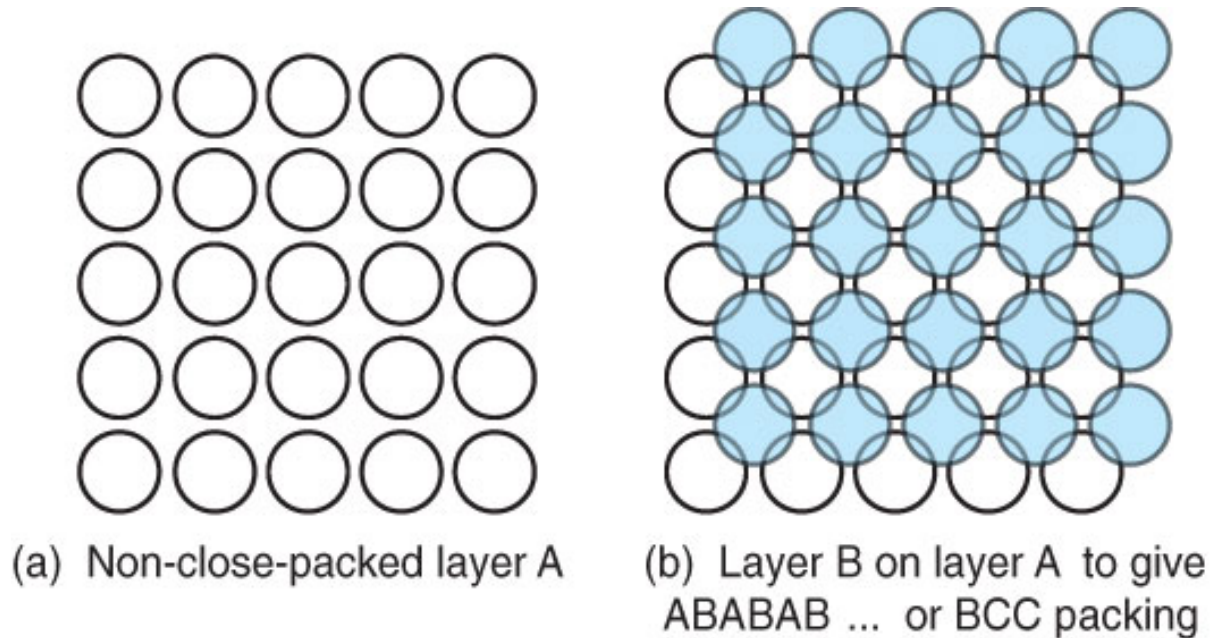
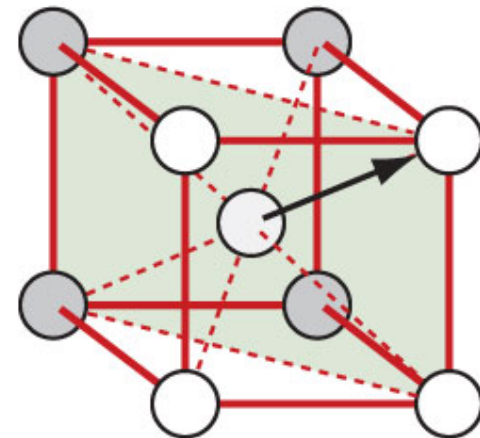


Figure 4.9 (a) A square grid of spheres; less efficient packing than FCC or HCP. (b) A second layer, B, nesting in the first; repeating this sequence gives ABAB . . . packing. If the sphere spacing is such that the gray spheres lie on the corners of a cube, the result is the non-close-packed BCC structure.

Pl question

Nature has many different ways of packing atoms. What is the main reason for all these differences?

1. Masses of the atoms
2. Diameters of the atoms
3. Force fields of the atoms
4. Valence of the atoms



Amorphous structure

- Not crystalline, disordered, lower packing fraction. 0.64 at best.

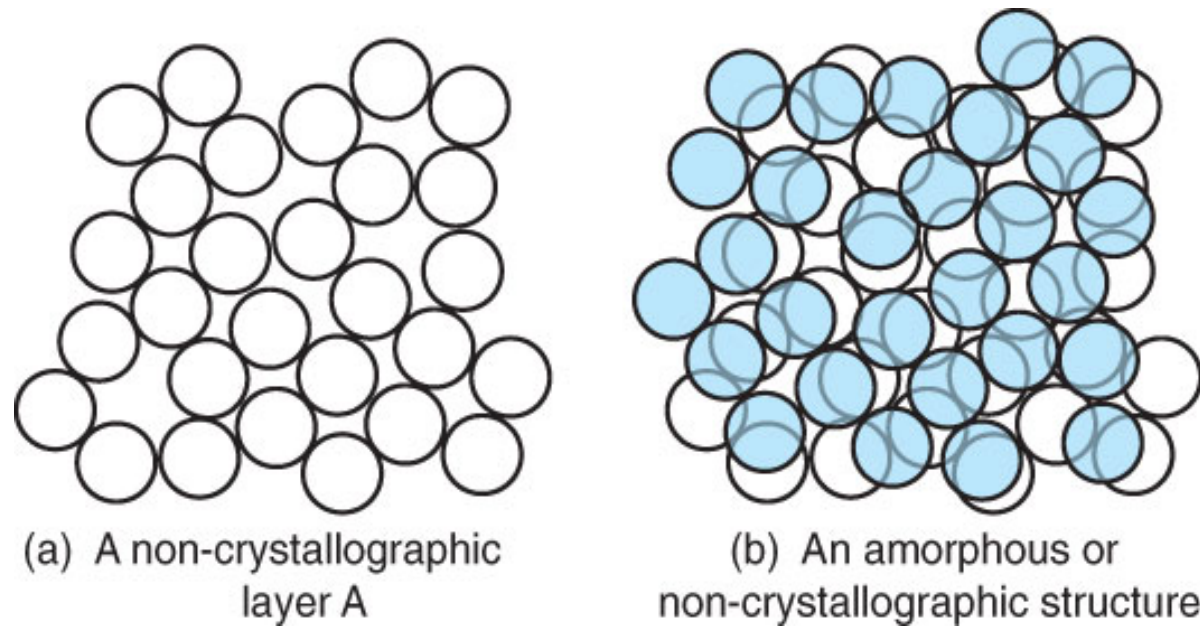


Figure 4.10 (a) An irregular arrangement of spheres in 2D.
(b) Extending this in 3D gives a random or amorphous structure.

Unit cells

- Crystalline materials are characterized by their unit cell.
- Most common: FCC, BCC, CPH (=HCP)

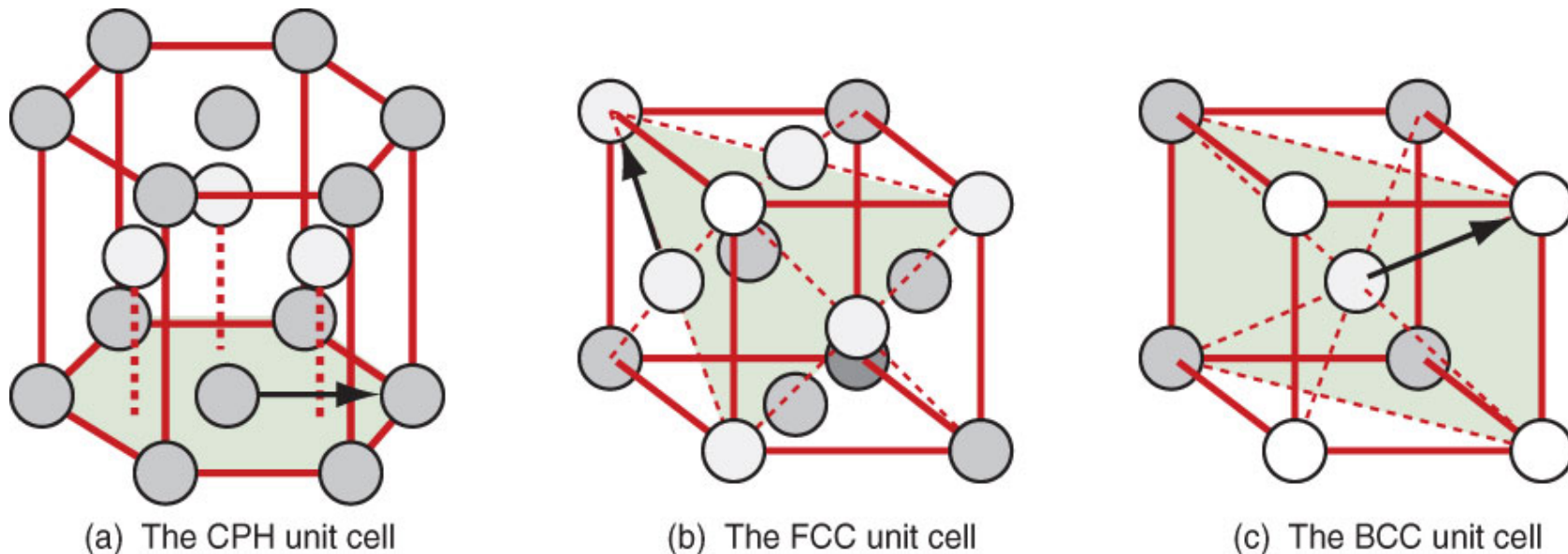


Figure 4.11 Unit cells. All the atoms are of the same type, but are shaded differently to emphasize their positions. (a) The close-packed hexagonal (CPH) structure. (b) The close-packed face-centered cubic (FCC) structure. (c) The non-close-packed body-centered cubic (BCC) structure. Arrows show nearest neighbors.

- Unit cells pack to fill space and create a *crystal lattice*; the points at which cell edges meet are called *lattice points*.

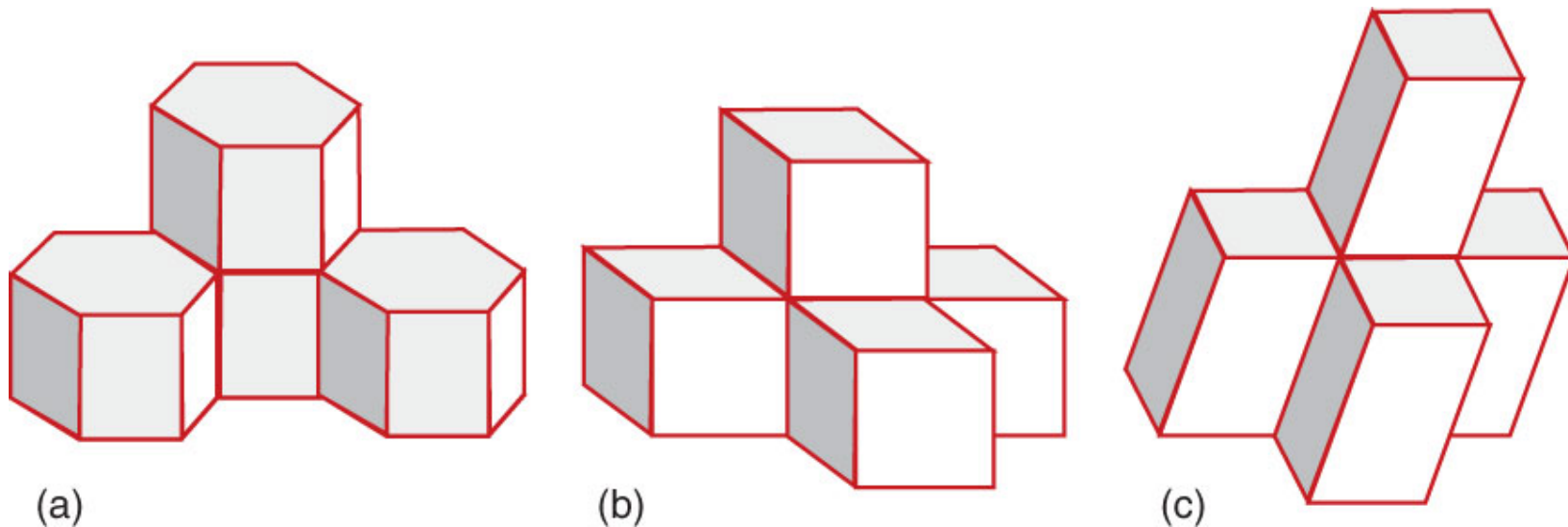
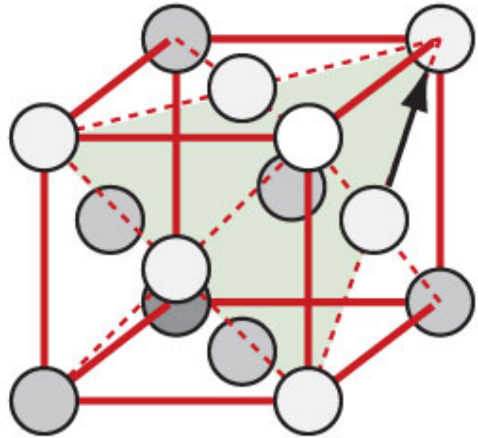


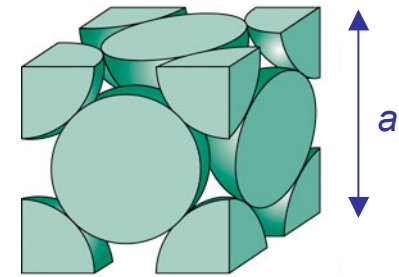
Figure 4.12 Unit cells stacked to fill space. (a) The hexagonal cell. (b) The cubic cell. (c) A cell with edges of differing length that do not meet at right angles.

Unit cells

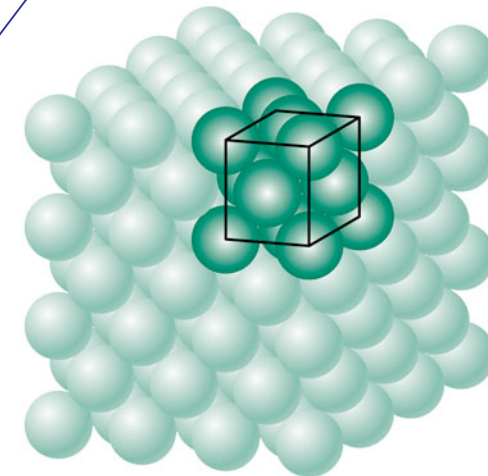


(b) The FCC unit cell

Careful: figures like these show atoms too small.
In reality they “touch” each other



(a)



(c)

FCC unit cell with real-sized atoms. The volume of the cube is a^3 .
We can count that the unit cell contains 4 full atoms.
The face diagonal ($a\sqrt{2}$) is equal to 2 atomic diameters or 4 atomic radii R .
From this we can calculate the packing fraction.
When we know a and the mass m , we can compute the density.

Unit cells

Solutions for packing fraction and density

$$4R = a\sqrt{2} \rightarrow R = a(2^{-3/2}).$$

$$V_{\text{atom}} = (4/3)\pi R^3 = (4/3)\pi a^3(2^{-9/2}).$$

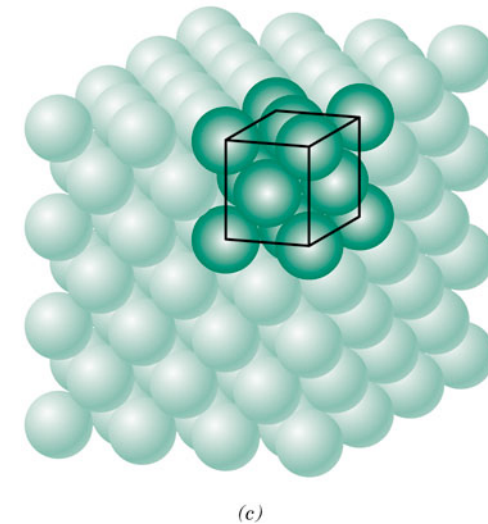
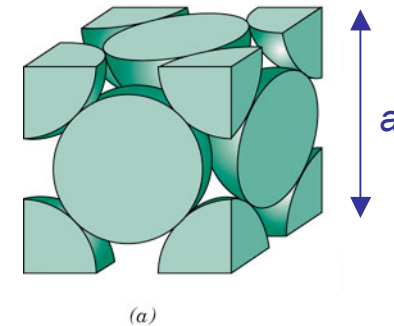
$$V_{4 \text{ atoms}} = (16/3)\pi R^3 = (16/3)\pi a^3(2^{-9/2}).$$

$$V_{\text{cube}} = a^3.$$

$$\text{Packing fraction} = V_{4 \text{ atoms}} / V_{\text{cube}} = (16/3)\pi(2^{-9/2}) = 0.74.$$

Suppose $a = 0.400 \text{ nm}$ and $m_{\text{atom}} = 60 \text{ u} = 60 \times 1.66 \times 10^{-27} \text{ kg}$.

$$\text{Density } \rho = 4m_{\text{atom}}/a^3 = 6225 \text{ kg/m}^3.$$



FCC unit cell with real-sized atoms. The volume of the cube is a^3 .

We can count that the unit cell contains 4 full atoms.

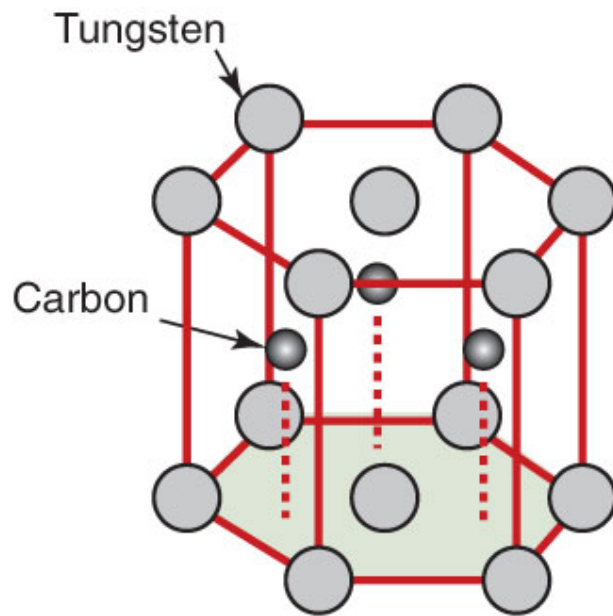
The face diagonal ($a\sqrt{2}$) is equal to 2 atomic diameters or 4 atomic radii R .

From this we can calculate the packing fraction.

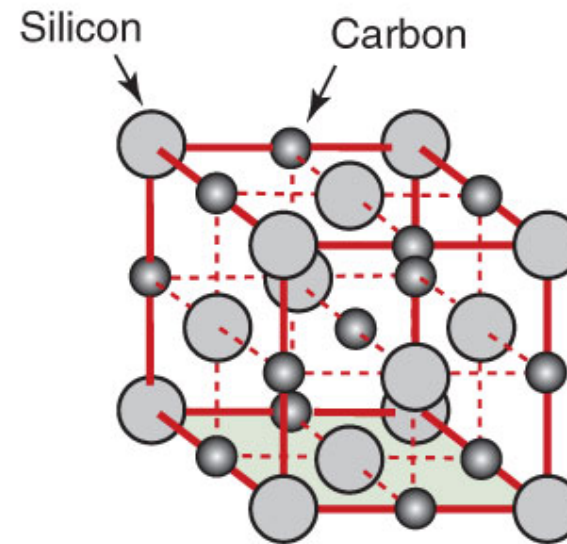
When we know a and the mass m , we can compute the density.

*Atom packing in **ceramics***

- Generally compounds made of two or more elements (A and B)
- A pair of atoms (A+B) is associated with each lattice point in the structures below.



(a) The tungsten carbide unit cell



(b) The silicon carbide unit cell

Figure 4.13 Unit cells of compounds. (a) Tungsten carbide. (b) One form of silicon carbide.

*Atom packing in **glasses***

- Compounds often resist crystallization on freezing and remain amorphous (some metal alloys also can behave this way when rapidly cooled – amorphous metallic ‘glasses’).
- Silica (SiO_2) is commonly amorphous and forms the basis of almost all glasses (mixed with Na_2O – soda glass; with B_2O_5 – Pyrex).

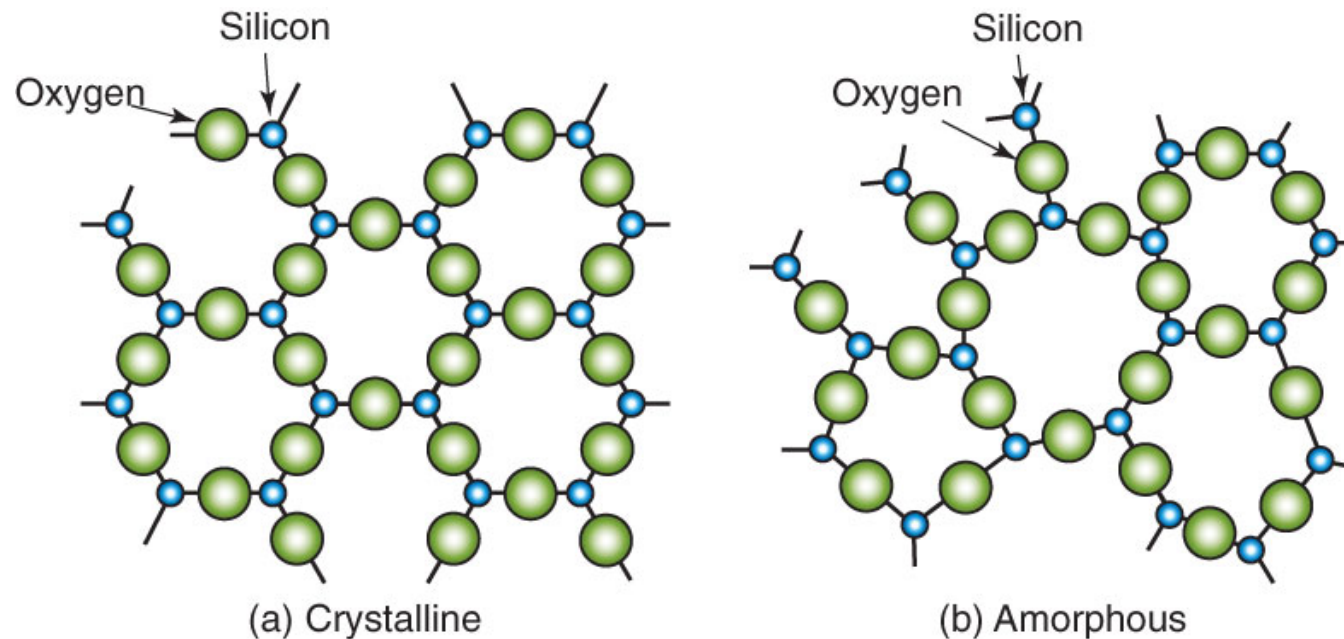


Figure 4.14 Two alternative structures for silica, the basis of most glasses.

(a) Crystalline silica. (b) Glassy or amorphous silica.

*Atom packing in **polymers***

- The backbone of a high molecular weight polymer is a long chain of carbon atoms to which side groups are attached.
- The simplest is polyethylene, PE, $(-\text{CH}_2-)_n$. The chains have ends, here CH_3 . PE is made by polymerization (=snapping together) of ethylene molecules, $\text{CH}_2=\text{CH}_2$; the double bond is broken allowing links to carbon neighbors on both sides.

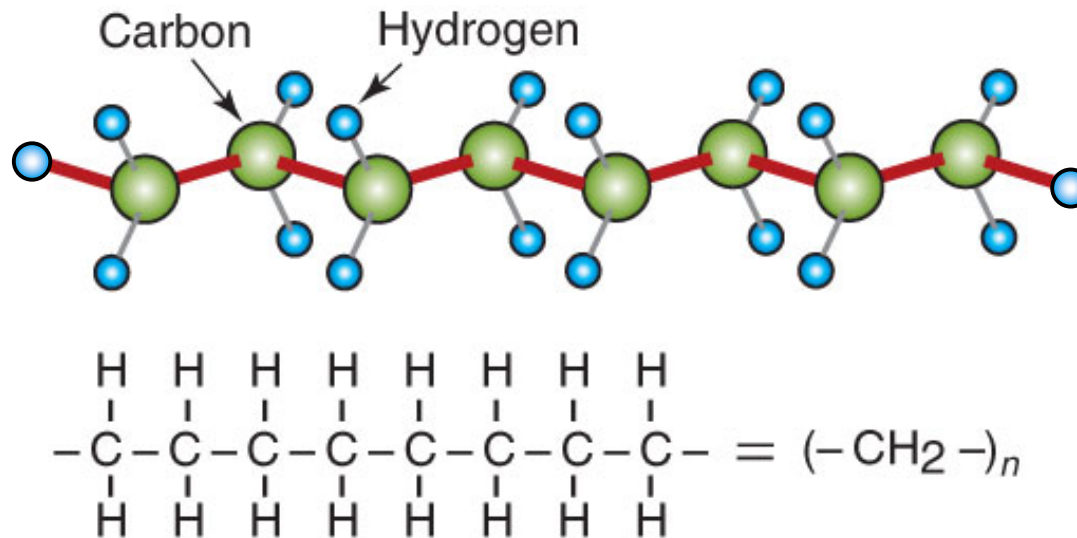
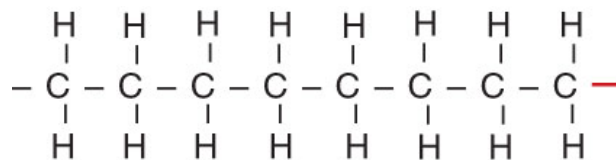
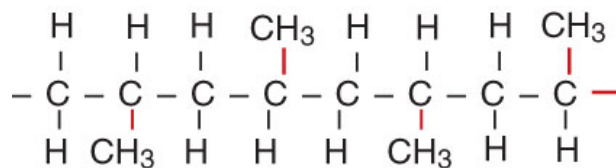


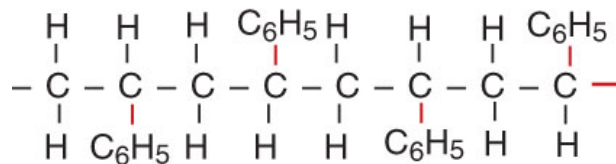
Figure 4.15 Polymer chains have a carbon–carbon backbone with hydrogen or other side groups. The figure shows three alternative representations of the polyethylene (PE) molecule.



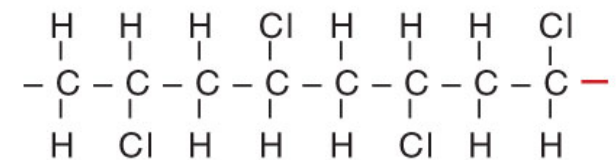
Polyethylene, **PE**



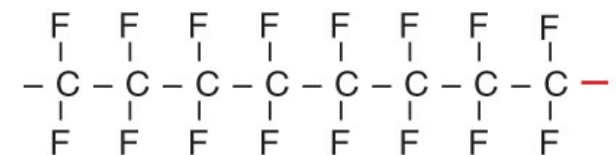
Polypropylene, **PP**



Polystyrene, **PS**



Polyvinyl chloride, **PVC**



Polytetrafluoroethylene, **PTFE**



Figure 4.16 Five common polymers, showing the chemical make-up. The strong carbon–carbon bonds are shown in red.

- Polymer molecules bond together to form solids; the chains of a linear polymer (10^3 to 10^6 units in length): strongly bonded along the chain.
- If separate chains attract each other via weak 'hydrogen' bonds we get an amorphous thermoplastic. Melts easily and moldable.

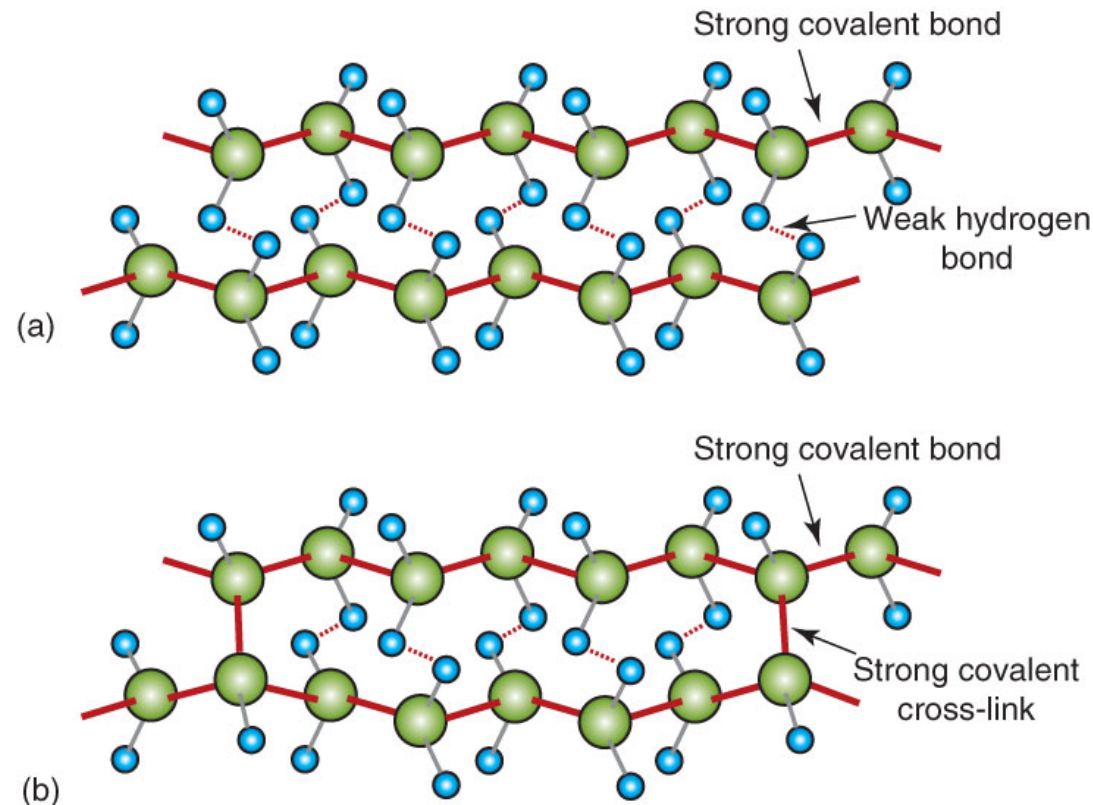


Figure 4.17 (a) Polymer chains have strong covalent 'backbones', but bond to each other only with weak hydrogen bonds unless they become cross-linked. (b) Cross-links bond the chains tightly together.

The strong carbon-carbon bonds: solid red lines.

- In local regions of good chain alignment the hydrogen bonds can produce crystallites (1-10 μm) which scatter light.

Transparent polymers like PC, PMMA, PS have no crystallites (are amorphous), while PE and nylon (PA) are translucent.

- Replacing hydrogen bonds with stronger covalent C-C bonds (**cross-linking**) produces important changes in properties.

Elastomers have a few such cross-links and **thermosets** have **many**. Covalent cross-links are not broken by heating – not thermally molded and difficult to recycle.

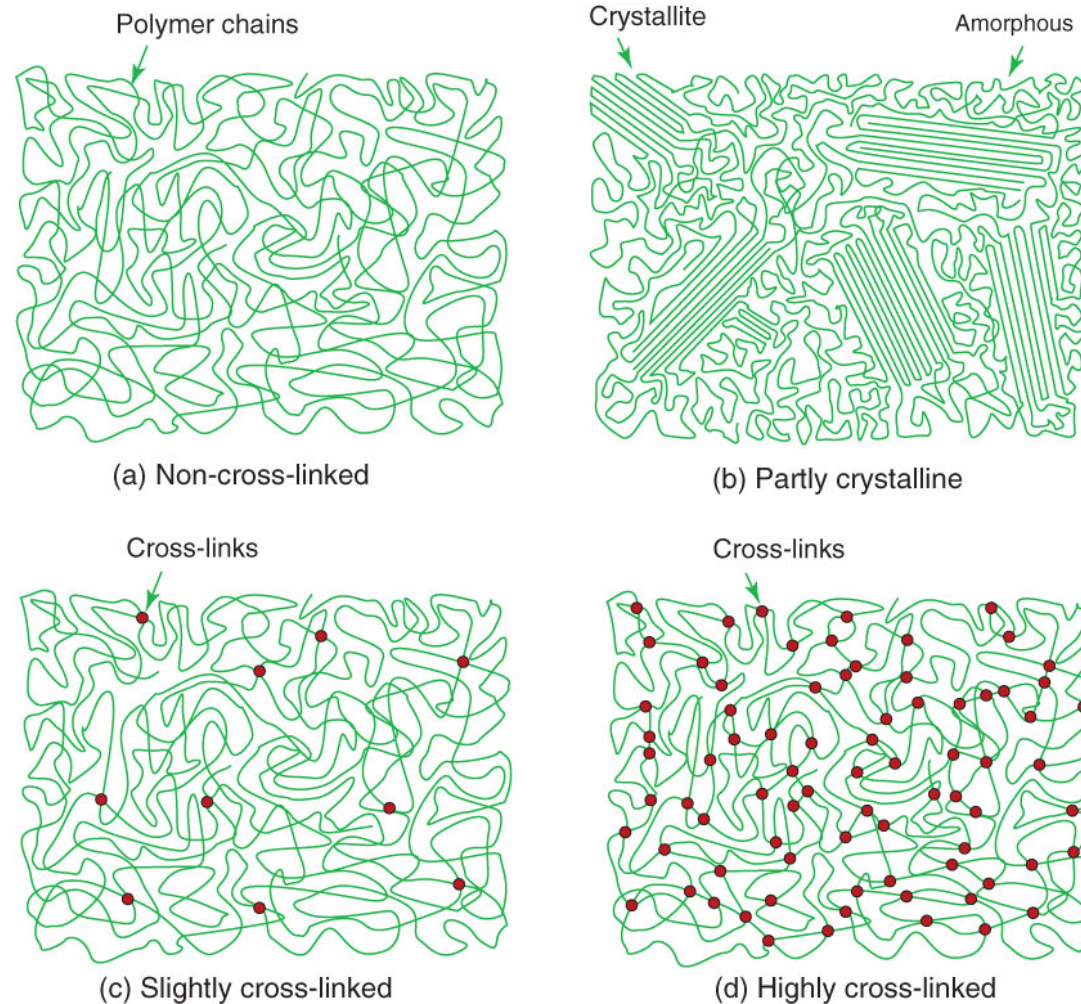


Figure 4.18 (a) Chains in polymers like PP form spaghetti-like tangles with no regular repeating pattern - amorphous or 'glassy'. (b) Some polymers form regions in which the chains line up and register, giving crystalline patches - partly crystalline polymer structure. (c) Elastomers have occasional cross-links between chains, but these are far apart, allowing the chains between them to stretch. (d) Heavily cross-linked polymers like epoxy inhibit chain sliding.

Cohesive energy and elastic moduli: crystals and glasses

- **Cohesive energy** – energy per mol to separate atoms of a solid completely; higher cohesive energy – higher modulus (strong and stiff bonds).
- Bond energy is minimum at equilibrium spacing, a_0 . A spring stretched by δ under a force F has a stiffness $S = F/\delta$.
- Relating stiffness to modulus: pulling two atoms apart by δ gives a strain of $\varepsilon = \delta/a_0$. The stress is given by $\sigma = F/a_0^2$ (force/area). Thus $\sigma = S\varepsilon/a_0$.
- From $\sigma = E\varepsilon$ (Hooke's Law): $E = S/a_0$.
- For a large atom with a weak bond ($S \sim 0.5$ N/m) we get a lower limit of ~ 1 GPa for E for a true solid (exceptions: foams and elastomers).

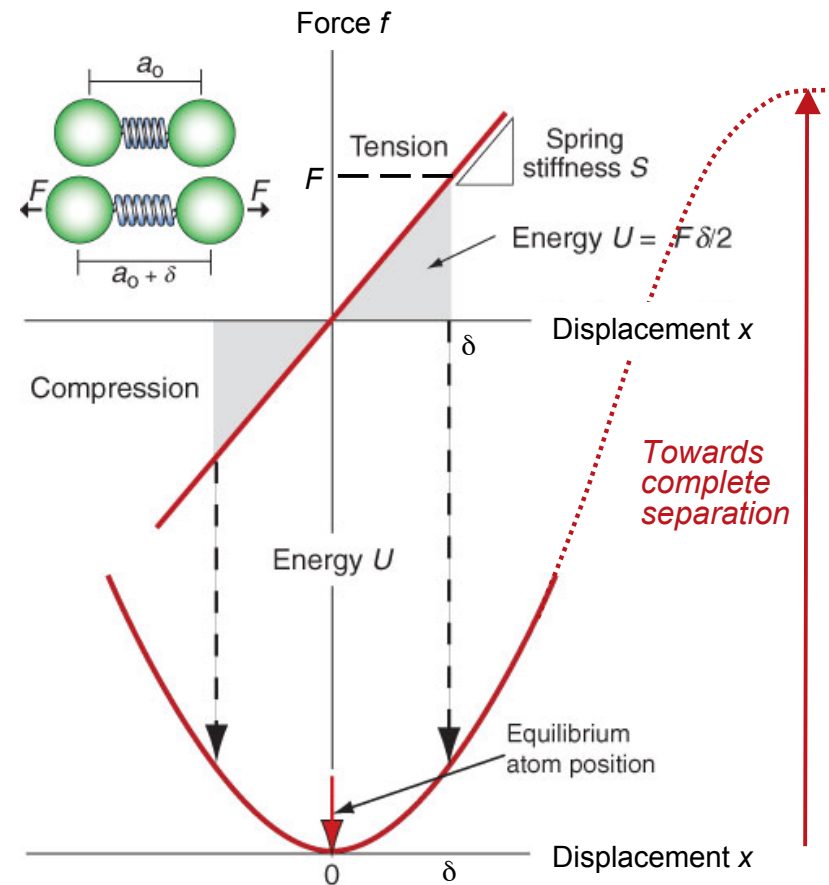


Figure 4.19

Atoms in solids are linked by bonds that behave like springs. Bond stiffness $S = F/\delta$. Stretching or compressing the bond by displacement δ stores energy $U = \int_0^{\delta} f(x) dx = F\delta/2$. The equilibrium atom separation is at the bottom of the energy well.

Table 4.1 Bond stiffnesses

Bond type	Examples	Stiffness, S (N/m)	Young's modulus, E (GPa)
Covalent	Carbon-carbon	50 – 180	200 – 1000
Metallic	All metals	15 – 75	60 – 300
Ionic	NaCl	8 – 24	32 – 96
Hydrogen bond	PE	3 – 6	2 – 12
Van der Waals	Waxes	0.5 – 1	1 - 4

Question

How large is the atomic distance a_0 in these five cases?

$$\text{Young's modulus } E = S/a_0$$

$$\text{Volume occupied by an atom } V_{\text{atom}} = a_0^3$$

The elastic moduli of elastomers

- Consider the bonds between chains (except for cross-links) to be very weak at room temperature (they have 'melted') – the glass temperature, T_g , is below RT (=room temperature). Segments freely slide over one another and the material would be a viscous liquid if not for the cross-links.
- Stretching produces local alignment (strain induced ordering).

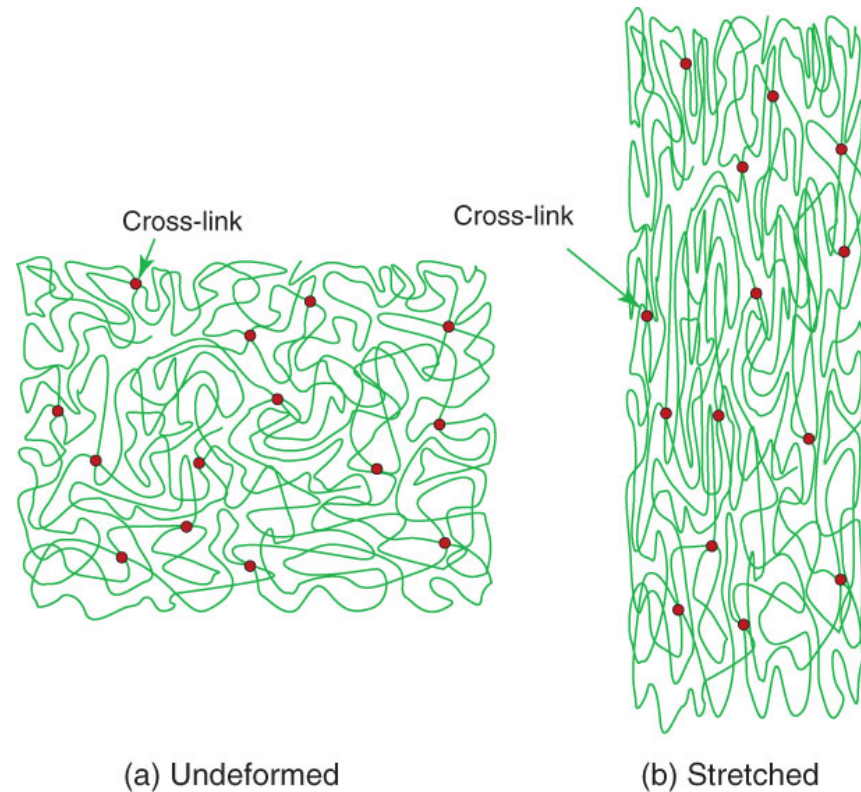


Figure 4.20 The stretching of an elastomer. Here the structure has been stretched to twice its original length. The **stretching causes alignment**, producing crystal-like regions. Thermal vibration drives the structure back to the one on the left, restoring its shape.

Mixtures of atoms

- Solid solution – like sugar dissolved in tea, but in a solid.
- Many properties are changed significantly by making a solid solution (as we will see) but density and modulus are not.
- Rule of mixtures for density:

$$\rho^* = f\rho_A + (1 - f)\rho_B \quad (\text{Vegard's law}),$$

f is the fraction of A atoms.

- To influence density and modulus we can turn to composites and foams (hybrid materials). Next.

4.5 Manipulating the modulus and density

- Composites: embedding fibers or particles in a continuous matrix of a polymer (PMC), metal (MMC) or ceramic (CMC).
- Composites can have high stiffness and strength per unit weight and for MMCs and CMCs good high T performance.
- For reinforcement r and matrix m the rule of mixtures is exact for density:
$$\rho^* = f\rho_r + (1 - f)\rho_m \quad (\text{Eq. 4.20})$$
 f is volume fraction.

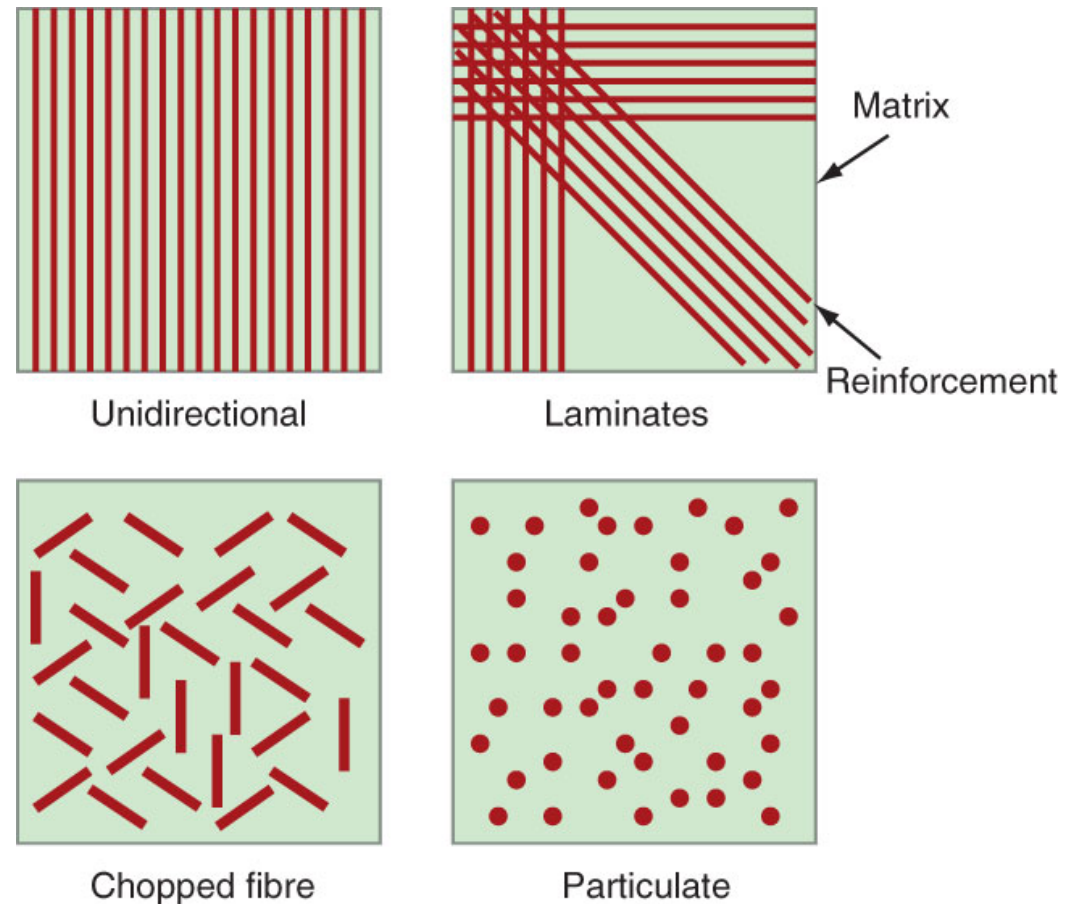


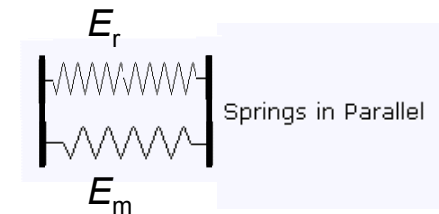
Figure 4.21 Manipulating the modulus by making composites, mixing stiff fibers or particles into a less-stiff matrix.

The modulus of a composite is bracketed by 2 bounds.

- The upper limit, E_U^* , assumes the two components **strain** equally (springs in parallel) giving:

$$E_U^* = fE_r + (1 - f)E_m$$

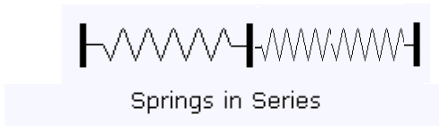
(Eq. 4.21)



- The lower bound, E_L^* , assumes the two components carry the same **stress** (springs in series) giving:

$$E_L^* = E_m E_r / [fE_m + (1 - f)E_r]$$

(Eq. 4.22)



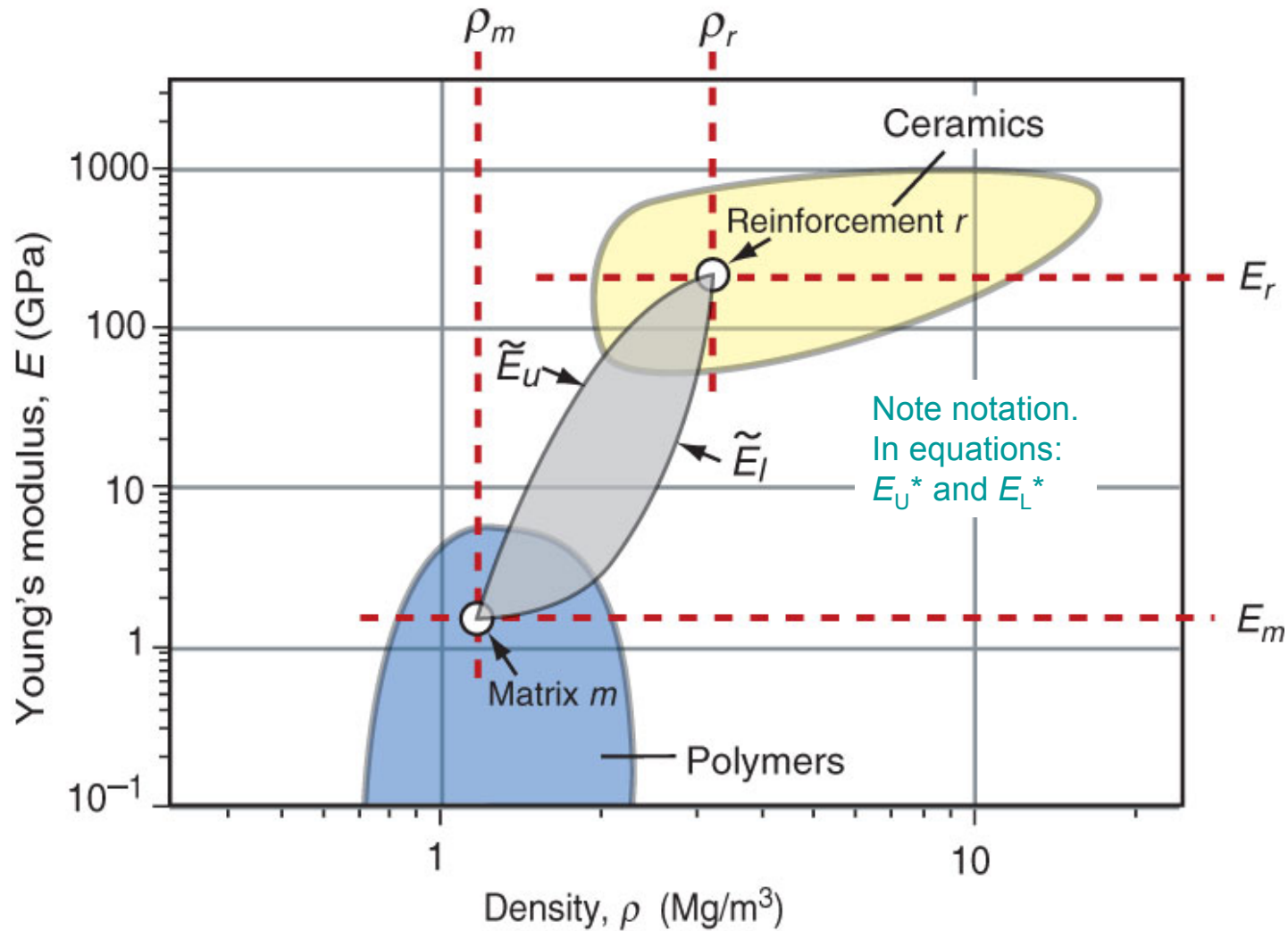
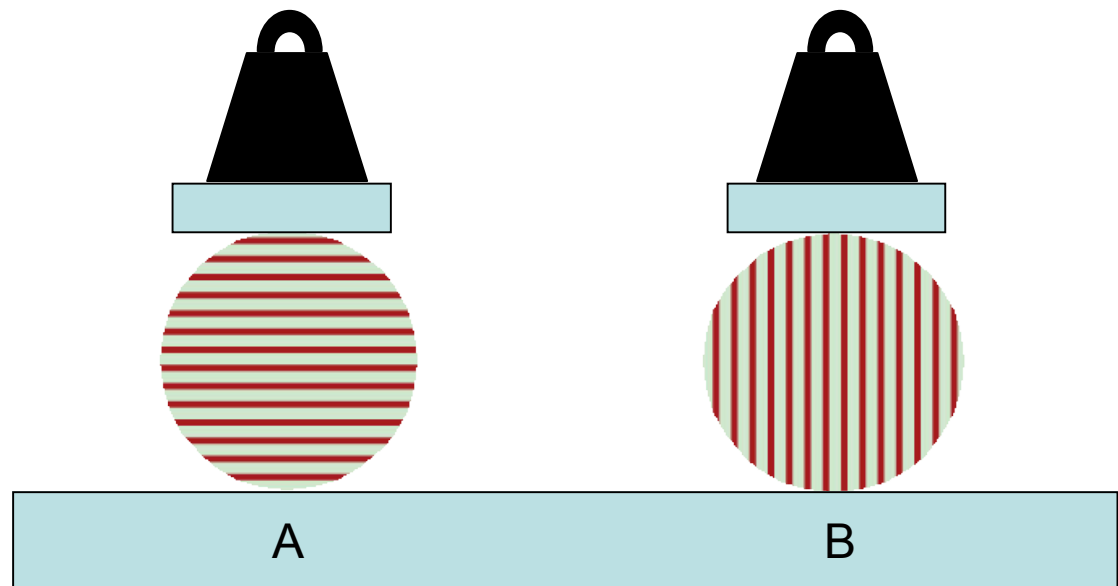


Figure 4.22 Composites made from a matrix m with a reinforcement r have moduli and densities, depending on the volume fraction and form of the reinforcement, that lie within the gray shaded lozenge bracketed by equations (4.21) and (4.22). Here the matrix is a polymer and the reinforcement a ceramic, but the same argument holds for any combination.

PI question

In which case is the wheel compressed most?

1. A
2. B
3. Equally far



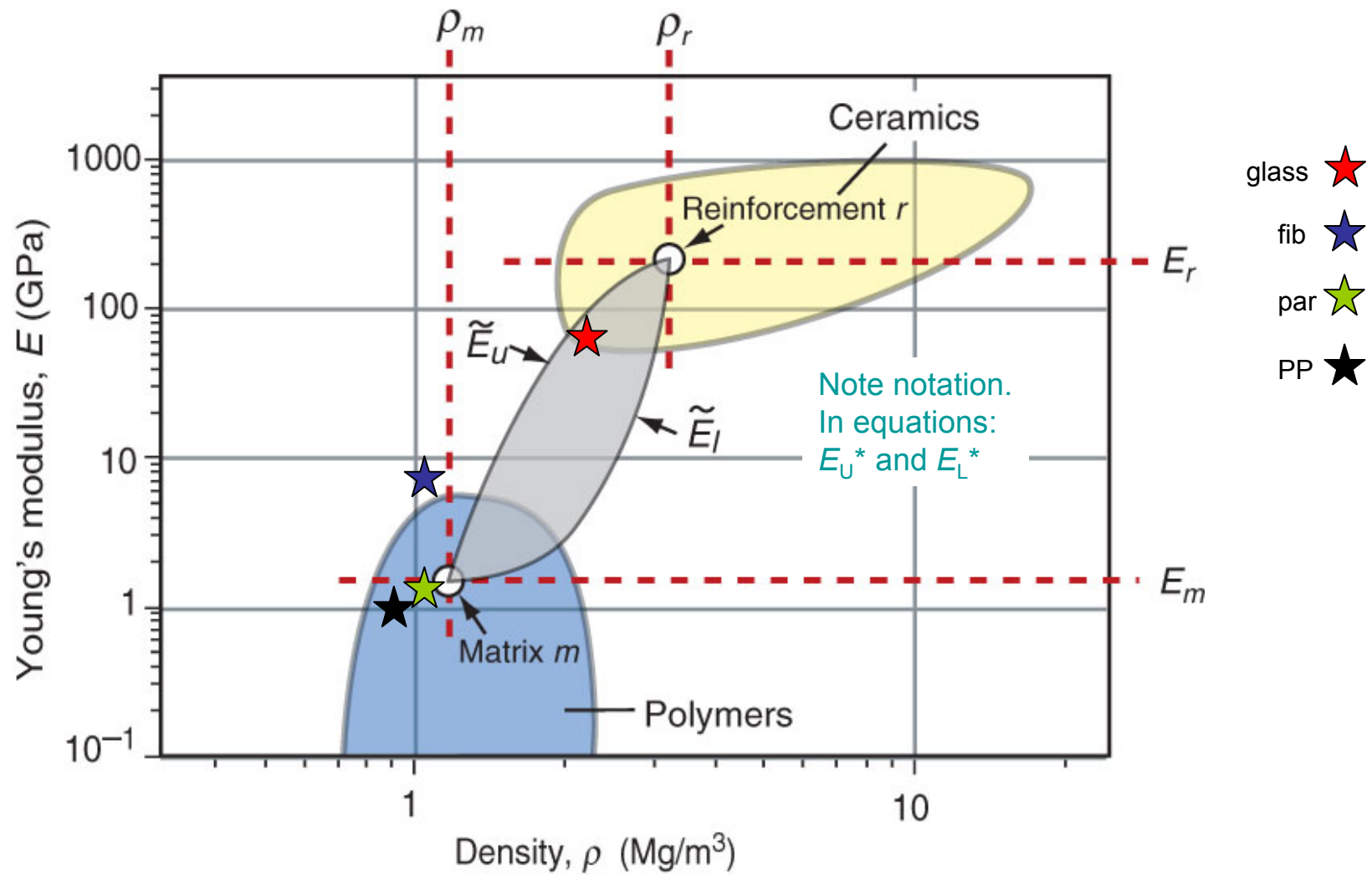
Example

Example 4.4

A composite material has a matrix of polypropylene and contains 10% glass reinforcement. Estimate the Young's modulus and density of the composite: (i) if the glass is in the form of long parallel fibres (calculate the properties parallel to the fibres); (ii) if the glass is in the form of small particles. Compare the specific moduli (E/ρ) of the constituent materials and the two composites and plot the four materials on a copy of Figure 4.22.

Use the following property data:

	E (GPa)	ρ (Mg/m ³)	E/ρ
Polypropylene (m)	1	0.9	1.1
Glass (r)	70	2.2	31.8



Foams

- Polymer foams are common (insulation, flotation, cushions, packaging) but metal, ceramic, and even glass foams are now becoming important.
- Light, low modulus.
- Cellular solid is characterized by relative density = fraction of foam occupied by solid, $\rho^*/\rho_s = (t/L)^2$ ($t \ll L$)
3rd direction (vertical) is not reduced by (t/L)

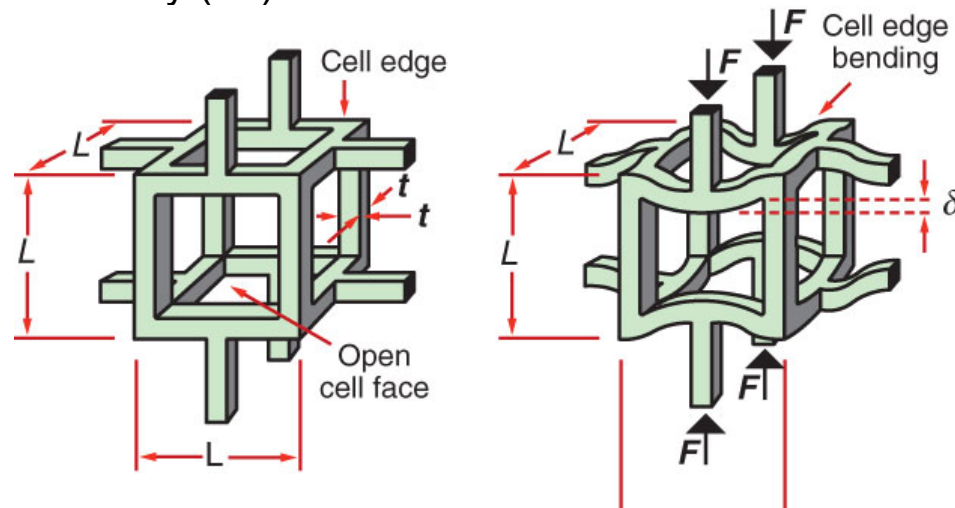


Figure 4.23 Manipulating the modulus by making a foam—a lattice of material with cell edges that bend when the foam is loaded. Closed cell faces are also possible.

- When foam is loaded the cell walls bend and the behavior can be modeled to give the foam modulus:

$$E^*/E_s = (\rho^*/\rho_s)^2 \quad (\text{Eq. 4.24})$$

- Allows for **very low modulus materials**

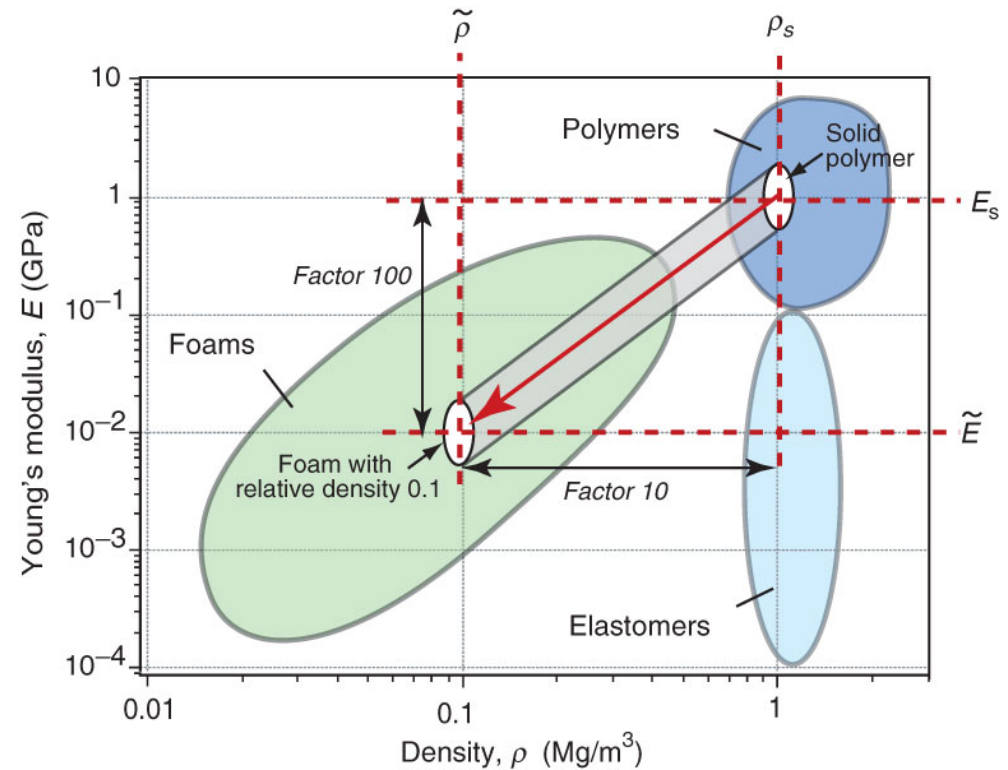


Figure 4.24 Foaming creates new materials with lower modulus and density. Low modulus is good for making packaging and protective shielding; low density is good for lightweight design and for flotation. The red arrow is a plot of equation (4.24).