Chapter 5
Flex, sag and wobble: stiffness-limited design

5.1 Introduction and synopsis

- Strength is seen to be critical, but stiffness is often taken for granted. E.g. London Millennium Bridge, Tacoma Narrows Bridge
- Real loading situations can be decomposed into the common modes of tension, compression, bending, and torsion.
- Here we explore standard solutions to **elastic** problems, use them to derive material limits and indices, plot them onto material property charts, and finally review case studies.

5.2 Standard solutions to elastic problems

I. Extension or compression

II. Bending of beams

III. Torsion of shafts

IV. Buckling of columns and plates

V. Vibrating beams and plates

I. Elastic extension or compression

- $\sigma = F/A$; $\varepsilon = \sigma/E$; $\varepsilon = \delta/L_0$; thus the relation between load F and deflection δ is

$$\delta = L_0 F/AE$$

and the stiffness S (not the same as E itself) is defined as

$$S = F/\delta = AE/L_0 \qquad \text{(Eq. 5.2)}$$

the *shape* of the cross-sectional area does not matter

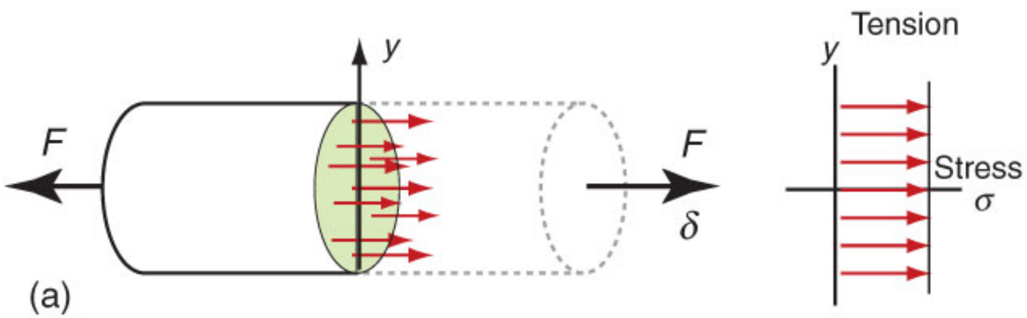
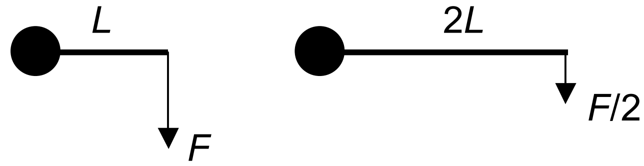


Figure 5.1 (a) A tie with a cross-section *A* loaded in tension. Its stiffness is $S = F/\delta$.

“Moment” = “Torque” = (Force) x (distance from the center)



II. Elastic bending of beams

- A beam loaded by a bending moment *M* has its axis deformed to curvature $\kappa = d^2u/dx^2$, *u* is the displacement parallel to the *y*-axis.
- Curvature generates a linear variation of strain (and stress), tension (+) on one side, compression (–) on the other
- Beam theory: the stress profile caused by a moment *M* is given by

$$\frac{\sigma}{y} = \frac{M}{I} = E\kappa = E\frac{d^2u}{d^2x}$$

with *I* the second moment of area (next page)

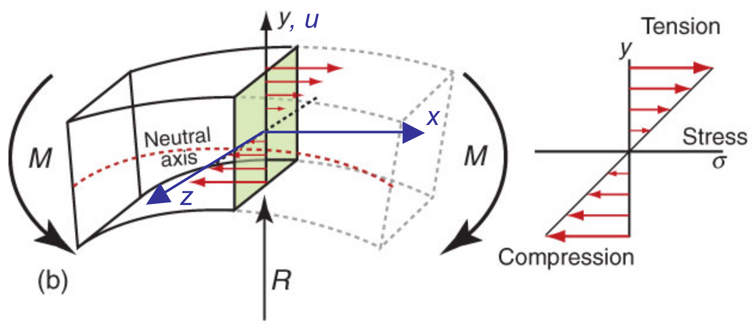
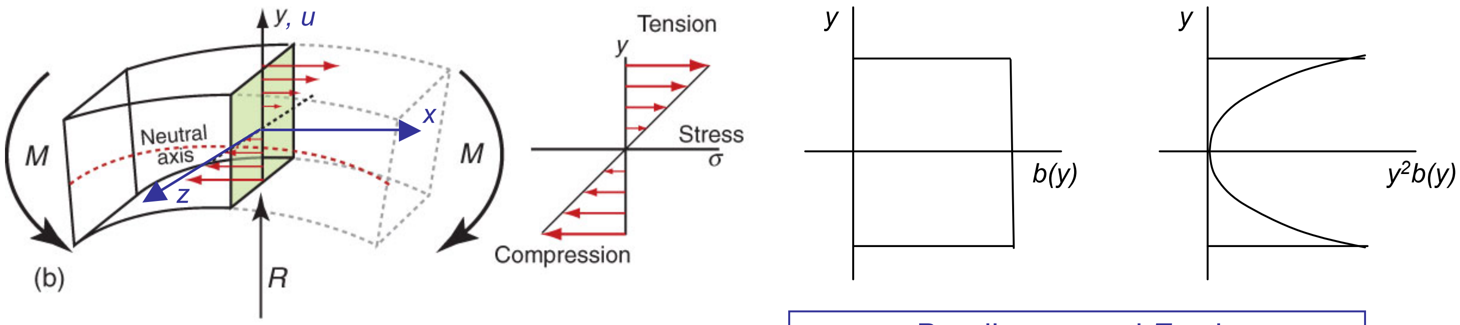


Figure 5.1 (b) A beam of rectangular cross-section loaded in bending. The stress σ varies linearly from tension to compression, changing sign at the neutral axis, resulting in a bending moment *M*. *R* is the radius of curvature.



Bending around Z axis

- I is the second moment of inertia: $I = I_{ZZ} = \int_{section} y^2 b(y) dy$;

  (Don’t confuse moment M with moment I)

  y is measured vertically from the neutral axis and b(y) is the width of the section at y (in z-direction)

  I characterizes the resistance to bending and depends on both size and shape.

- $M/\kappa = EI$ = the flexural rigidity, related to F/δ = stiffness

- The stiffness for a beam of length L with a transverse load F is

$$S = F/\delta = C_1 EI/L^3$$ (C_1 in Fig. 5.3, two pages further)

Example

Example 5.1 *Beware: Now the Z axis is called the X axis*

(a) A beam has a rectangular cross with height h and width b. Show that the second moment of area is $I = bh^3/12$.

(b) A steel ruler is 300 mm long with a width w = 25 mm and a thickness t = 1 mm. Calculate the second moments of area I_{XX} and I_{YY}.*(i.e. for bending around X axis and Y axis)*

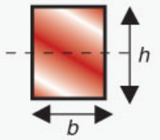
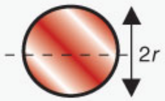
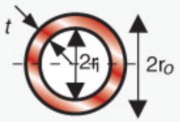
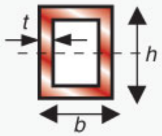
	Resistance to		
	stretching	bending	torsion (see later)
Section shape	**Area *A*** m^2	**Moment *I*** m^4	**Moment *K*** m^4
	bh	$\frac{bh^3}{12}$	$\frac{bh^3}{3}\left(1-0.58\frac{b}{h}\right)$ $(h > b)$
	πr^2	$\frac{\pi}{4}r^4$	$\frac{\pi}{2}r^4$
	$\pi(r_o^2 - r_i^2)$ $\approx 2\pi rt$	$\frac{\pi}{4}(r_o^4 - r_i^4)$ $\approx \pi r^3 t$	$\frac{\pi}{2}(r_o^4 - r_i^4)$ $\approx 2\pi r^3 t$
	$2t(h+b)$ $(h, b >> t)$	$\frac{1}{6}h^3 t\left(1+3\frac{b}{h}\right)$	$\frac{2tb^2h^2}{(b+h)}\left(1-\frac{t}{h}\right)^4$

Figure 5.2 Cross-section area and second moments of sections for four section shapes.

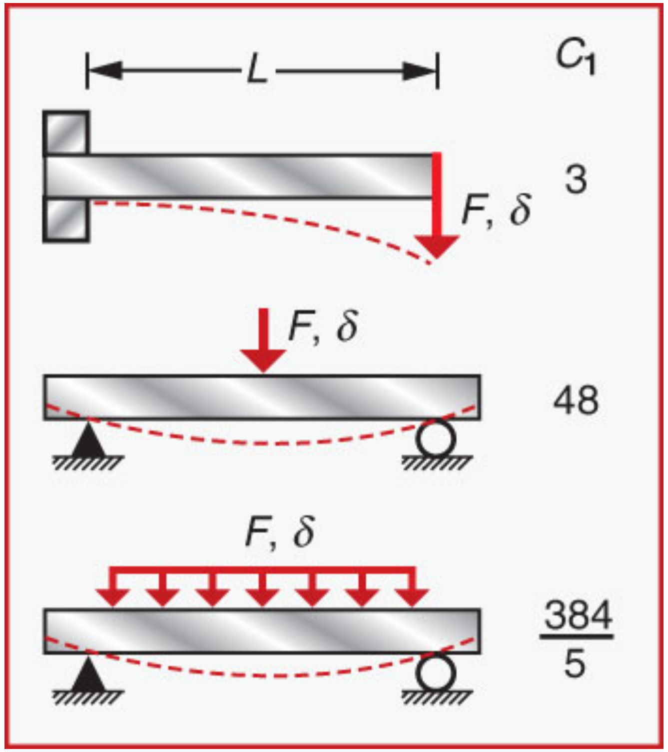


Figure 5.3 Elastic deflection of beams. The deflection δ of a span L under a force F depends on the flexural stiffness EI of the cross-section and the way the force is distributed. C_1 is defined in equation (5.5),

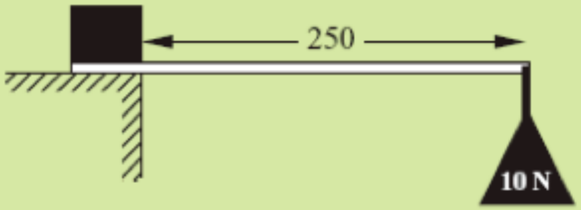
$$S = F/\delta = C_1 EI/L^3$$

Example

Example 5.2

(a) The ruler in Example 5.1 is made of stainless steel with Young's modulus of 200 GPa. A student supports the ruler as a horizontal cantilever, with 250 mm protruding from the edge of a table and hangs a weight of 10 N on the free end. Calculate the vertical deflection of the free end if the ruler is mounted with: (i) the X-X axis horizontal; (ii) the Y-Y axis horizontal. (Ignore the deflection caused by the self-weight of the ruler.)

Answer.



III. Torsion of shafts

- A torque, T, on an isotropic bar of uniform cross-section generates a shear stress, τ. For circular sections $\tau/r = T/K$ where K measures the resistance to twisting (analogous to I, resistance to bending).
- For circular sections K = J, where $J = \int_{section} 2\pi r^3 dr$; the polar second moment of area (yet another "moment"!) For other shapes K<J.
- The twist per unit length, θ/L obeys $\tau/r = T/K = G\theta/L$; G is the shear modulus; the torsional rigidity, $GK = T/\theta$.

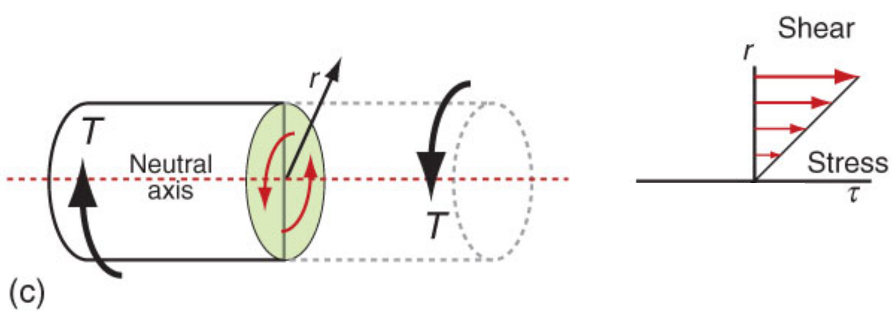


Figure 5.1 (c) A shaft of circular cross-section loaded in torsion.

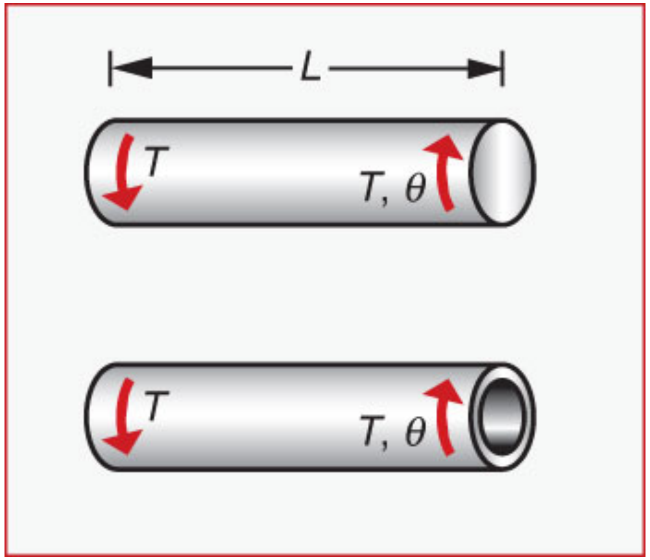


Figure 5.4 Elastic torsion of circular shafts. The stress in the shaft and the twist per unit length depend on the torque *T* and the torsional rigidity *GK*.

Example

Example 5.3

(a) Derive an expression for the polar second moment of area of a tube having a hollow circular section with inner radius r_i and outer radius r_o.

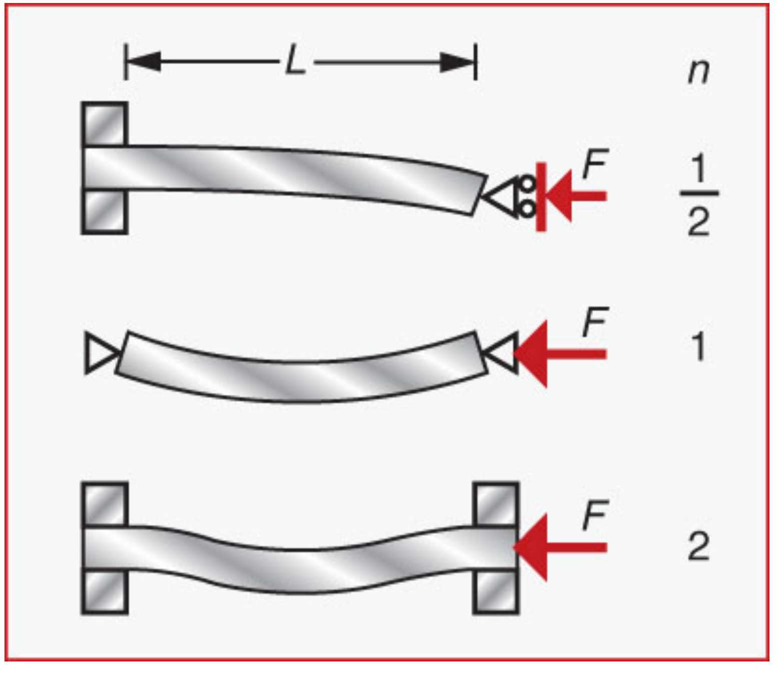
(b) A brass rod with shear modulus 40 GPa, length 200 mm, and having a solid circular cross-section with diameter 10 mm, is twisted with a torque of 10 Nm. What is the angle of twist?

Summary

	Stretching	Bending	Torsion
Load	F (force)	M (bending moment)	T (twisting torque)
Stress and effect	$\sigma = F/A = E\delta/L_0$	$\sigma/y = M/I = Ed^2u/dx^2$	$\tau/r = T/K = G\theta/L$
Resistance	Stiffness $S = AE/L_0$	Flexural rigidity EI	Torsional rigidity GK
Fig. 5.2	A/L_0	I	K

IV. Buckling of columns and plates

- A slender elastic column or plate can fail by buckling in compression at a critical load, F_{crit}, given by $F_{crit} = n^2\pi^2 EI/L^2$
- n depends on end constraints; it is the number of half wavelengths of the buckled shape. Slight misalignment can reduce F_{crit}.



Free to rotate and translate

Free to rotate

Clamped

Figure 5.5 The buckling load of a column of length *L* depends on the flexural rigidity *EI* and on the end constraints; three are shown here, together with the value of *n*.

V. Vibrating beams and plates

- Any undamped system vibrating at one of its natural frequencies reduces to a problem of a mass m attached to a spring of stiffness k;

  the lowest natural frequency of such a system is $f = (1/2\pi)(k/m)^{1/2}$.

- For common geometries and constraints $f = (1/2\pi)(C_2^2 EI/L^3 m)^{1/2}$ and thus (because $m = AL\rho$):

  $$f = (C_2/2\pi)(I/A)^{1/2}L^2(E/\rho)^{1/2}$$

Power 3: incorrect in book (Eq. 5.11)

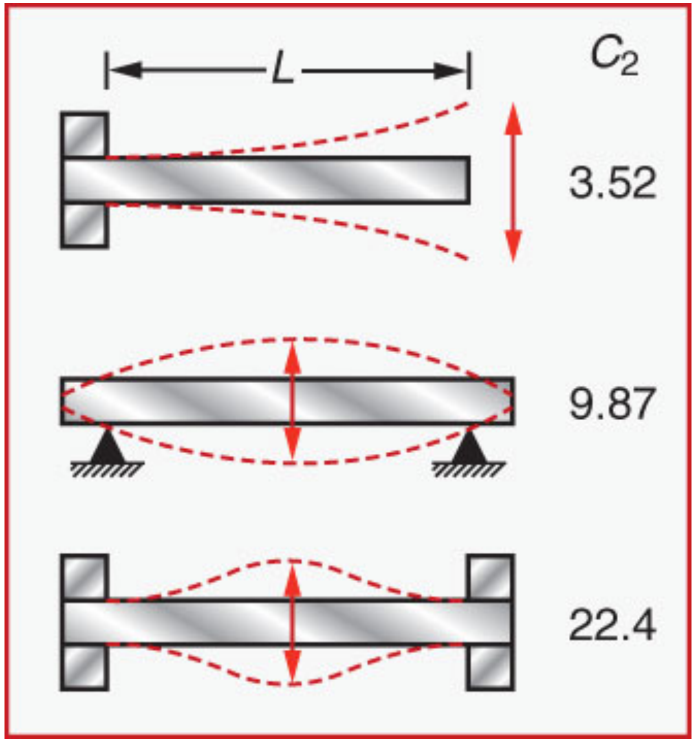
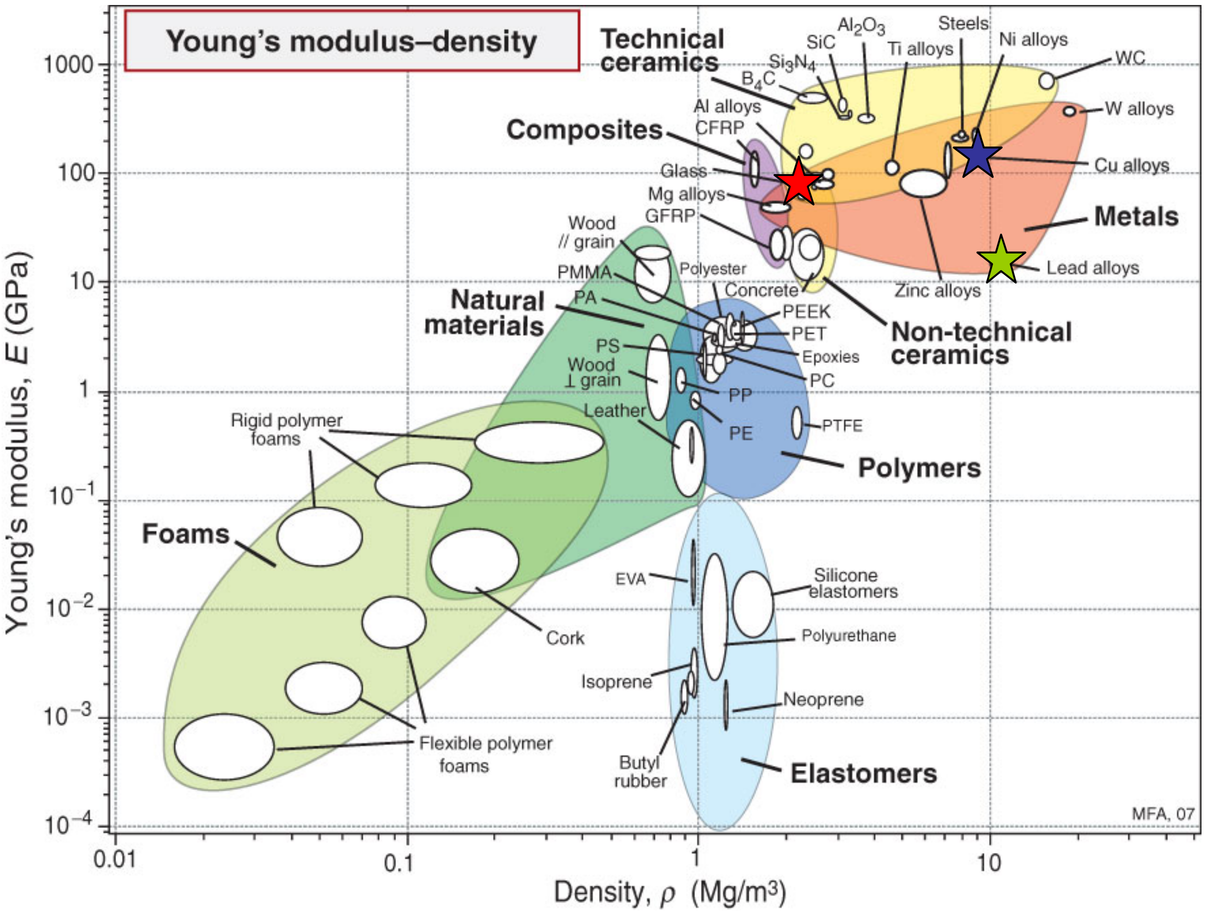


Figure 5.6 The natural vibration modes of beams clamped in different ways.

PI question

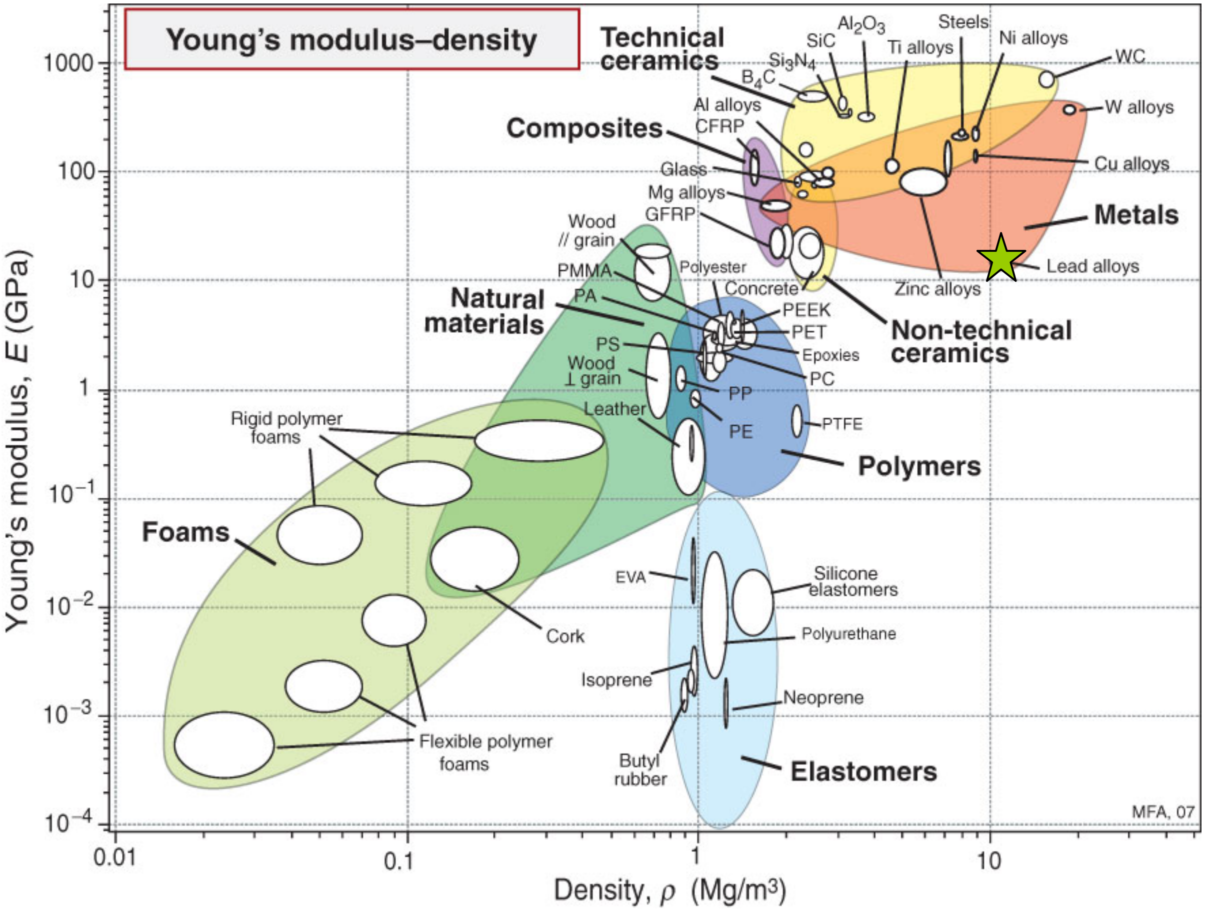
The speed of sound in a material is equal to √(*E*/ρ). In which material has the speed of sound the highest value?

1. ★Glass
2. ★Copper alloys
3. ★Lead alloys



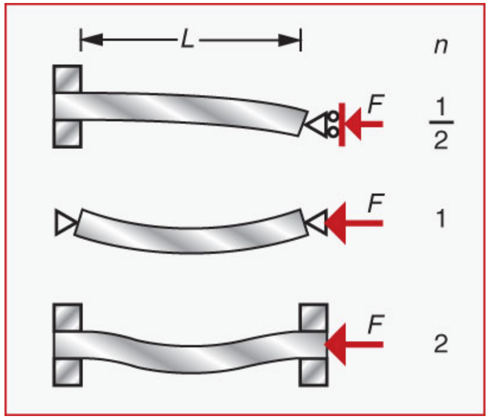
Exercise

How much is the speed of sound in lead alloys? ☆



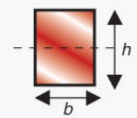
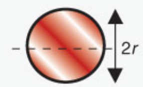
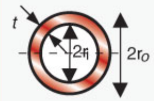
PI question

We compare the forces at which a clamped solid round rod and a clamped tube of the same length will begin to **buckle**. The rod and tube have equal outside diameters *r*.
For which wall thicknesses is the tube weaker than the rod?



$$F_{crit} = n^2\pi^2 EI/L^2$$

1. For all wall thicknesses
2. For thicknesses *t* smaller than *r*/4
3. For thicknesses *t* greater than *r*/4
4. The tube is never weaker than the rod

Section shape	Area *A* m²	Moment *I* m⁴
	bh	$\frac{bh^3}{12}$
	πr^2	$\frac{\pi}{4} r^4$
	$\pi(r_o^2 - r_i^2)$ $\approx 2\pi rt$	$\frac{\pi}{4}(r_o^4 - r_i^4)$ $\approx \pi r^3 t$

5.3 Material indices for elastic design

Let's implement the design process

- *Translation*
- *Screening*, based on constraints
- *Ranking*, based on objectives
- *Documentation* to give greater depth

Minimizing weight: a light, stiff tie-rod

- Constraints: length L_0, maximum extension δ^* at force F, thus stiffness at least $S^* = F/\delta^*$, reasonable toughness
- Objective: minimize mass
- Free variables:

  material, cross-sectional area A
- Objective function – equation describing the quantity to be maximized or minimized; here $m = AL_0\rho$

- Constraint: $S^* = AE/L_0$
- Eliminate free variable from objective function, here A: $m = S^* L_0^2(\rho/E)$
- S^* and L_0 are specified; the lightest tie-rod uses a material with the *smallest* ρ/E
- Invert to consider *maximum* values yielding material index $M_t = E/\rho$ – the specific stiffness

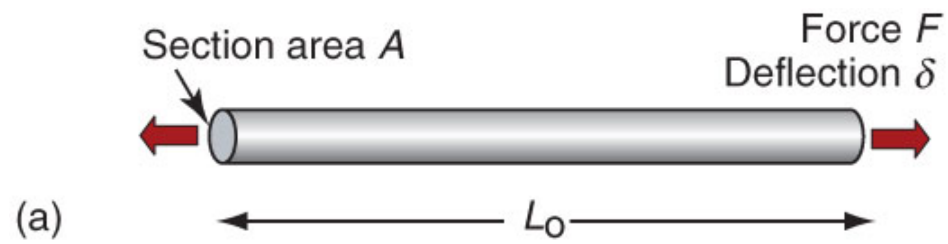
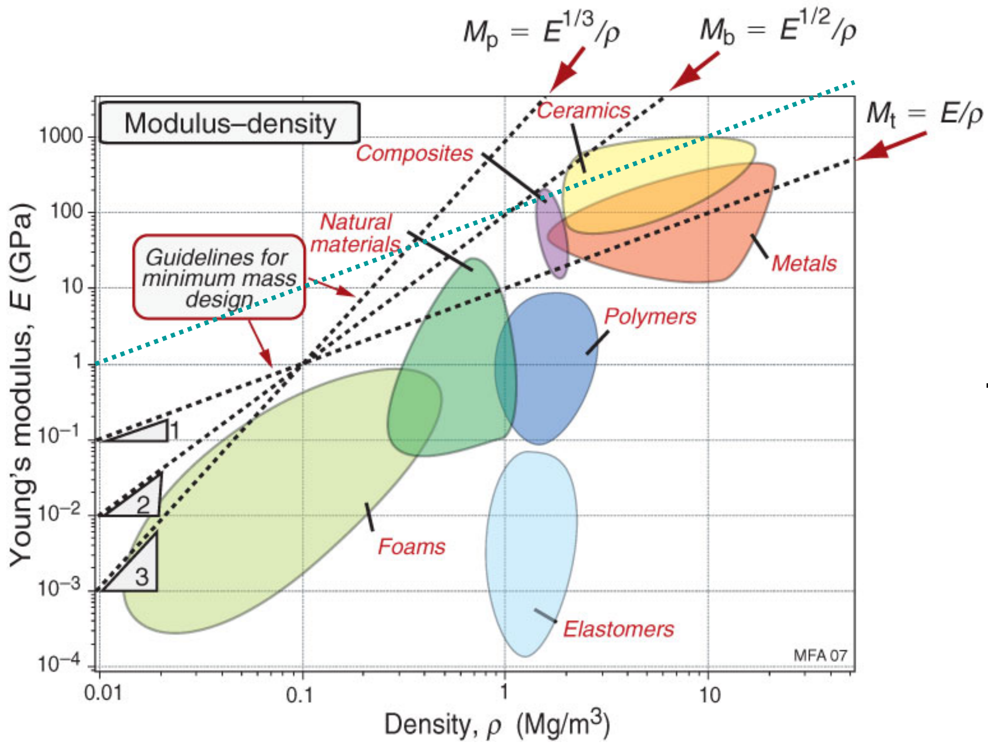


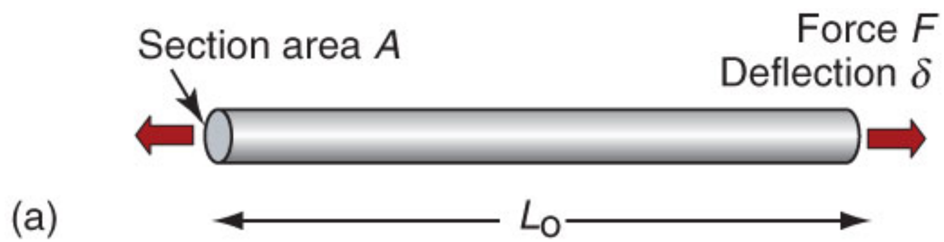
Figure 5.7 (a) A tie with cross-section area *A*, loaded in tension. Its stiffness is $S = F/\delta$ where *F* is the load and δ is the extension.

Minimizing weight: a light, stiff tie-rod



Choose material with *maximum* material index $M_t = E/\rho$ – the specific stiffness.

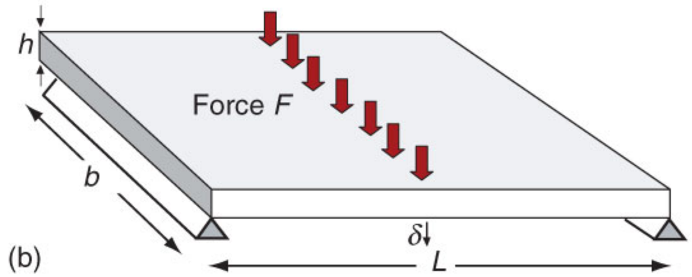
The line on the left has $E/\rho = 10$ GPa/(Mg/m^3). Shift line upwards for higher values.



Minimizing weight: a light, stiff panel

- Constraints: length, L, width b, maximum deflection δ, stiffness of S*
- Objective: minimize mass
- Free variables: material, thickness h
- Loaded in bending with a central load, F
- Objective function: $m = Al\rho = bhL\rho$
- Constraint: $S^* = C_1 EI/L^3$
- $I = bh^3/12$
- Eliminate free variable h: $m = (12S^*/C_1 b)^{1/3} bL^2 (\rho/E^{1/3})$
- S*, L, b, C_1 all specified; seek *smallest* $\rho/E^{1/3}$ or *maximize* the material index $M_p = E^{1/3}/\rho$

Figure 5.7 (b) A panel loaded in bending. Its stiffness is *S* = *F*/δ, where *F* is the total load and δ is the bending deflection.

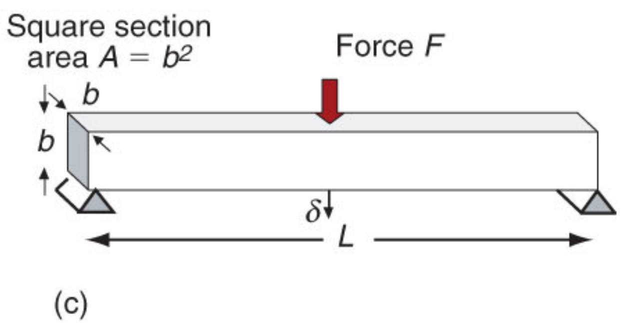


Minimizing weight: a light, stiff beam

- Constraints: the shape must be self-similar (all dimensions change in proportion as the overall size is varied, a simplification) length, L, square section, maximum deflection δ, stiffness of S*
- Objective: minimize mass
- Free variables: material, area of cross section
- Loaded in bending with a central load, F
- Objective function: $m = AL\rho = b^2L\rho$
- Constraint: $S^* = C_1EI/L^3$
- $I = b^4/12 = A^2/12$ [other shape: always a constant times A^2]
- Eliminate free variable b: $m = (12S^*L^3/C_1)^{1/2}L(\rho/E^{1/2})$
- S*, L, C_1 all specified; seek *smallest* $\rho/E^{1/2}$ or *maximize* the material index $M_b = E^{1/2}/\rho$; other shapes give the same answer, only with a different factor than 12.

Figure 5.7 (c) A beam of square section, loaded in bending. Its stiffness is $S = F/\delta$, where F is the load and δ is the bending deflection.

Factor 12: incorrect in book (p. 95, top)



By shaping the cross-section we can increase I without changing A by moving material away from the neutral axis (tubes, or I-beam), or decrease A without changing I.

- shape factor Φ: ratio of I for the shaped section to that for a solid square section with the same area (mass). Can't go too far – buckling.

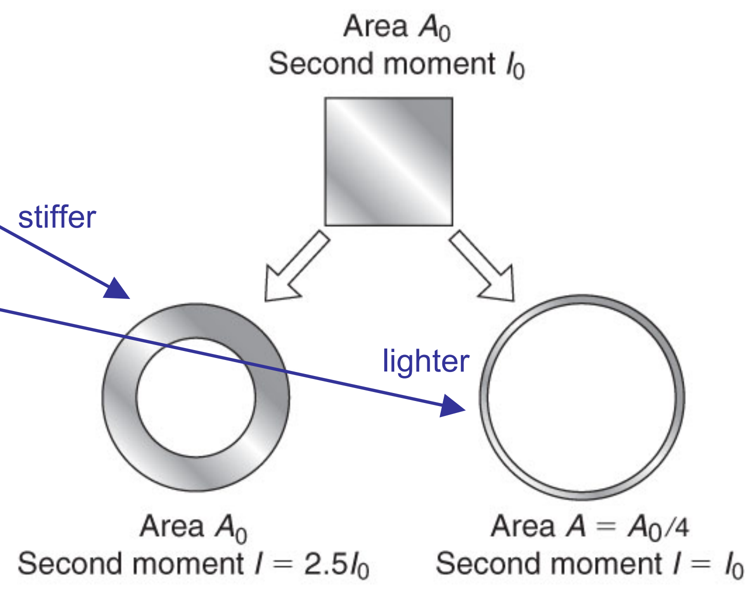


Figure 5.8 The effect of section shape on bending stiffness *EI*: a square-section beam compared with a tube of the same area (but 2.5 times stiffer) and a tube with the same stiffness (but four times lighter).

Material	Φ_{max}	Mass ratio
Steels	64	1/8
Al alloys	49	1/7
Composites	36	1/6
Wood	9	1/3

Minimizing material cost

- For material price C_m (\$/kg) the cost of material for a component of mass m is just mC_m.
- The objective function for the material cost C of the tie, panel, or beam becomes $C = mC_m = ALC_m\rho$
- Leads to indices as before replacing ρ with $C_m\rho$

Example

incorrect in book, p. 96, should be: **diameter**

solid rod, diameter 10 mm

Example 5.4

10 Nm

The brass rod in Example 5.3 is formed into a hollow tube with an outer radius of 25 mm, and the same length, using the same amount of material. It is twisted with the same torque. What is its angle of twist? Compare its torsional stiffness with the solid rod in Example 5.3.

Screening: attribute limits on charts

- Constraints can be plotted as horizontal or vertical lines; for example on the E-relative cost chart: E > 10 GPa; Relative cost < 3.

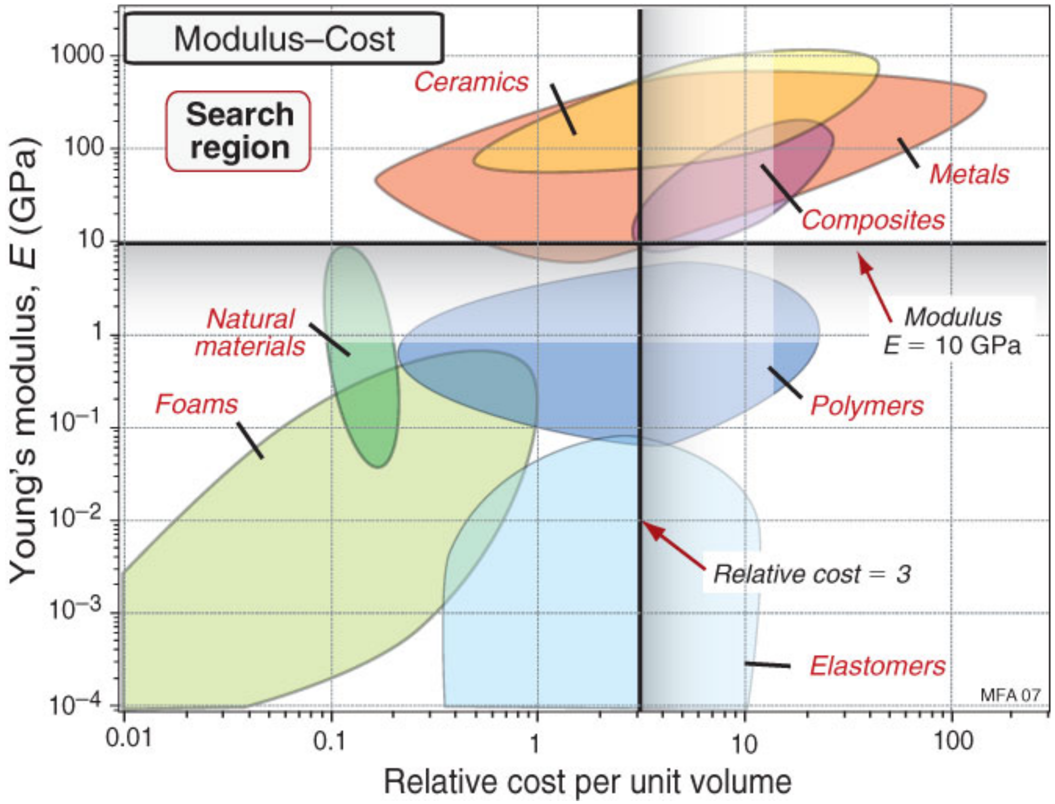


Figure 5.9 A schematic *E – Relative cost* chart showing a lower limit for *E* and an upper one for *Relative cost*.

***Ranking: indices on charts:* selection guidelines**

- Consider the design of light stiff components using the E-ρ chart
- Consider $M = E/\rho$ = constant, C (tie-rod)
- Take logs: $\log E = \log \rho + \log C$; a line of slope 1
- $M = E^{1/2}/\rho$ = constant, C (beam); $\log E = 2 \log \rho + 2 \log C$; a line of slope 2
- $M = E^{1/3}/\rho$ = constant C (plate); $\log E = 3 \log \rho + 3 \log C$; a line of slope 3
- All materials on a line perform equally well; those above are better, those below are worse.

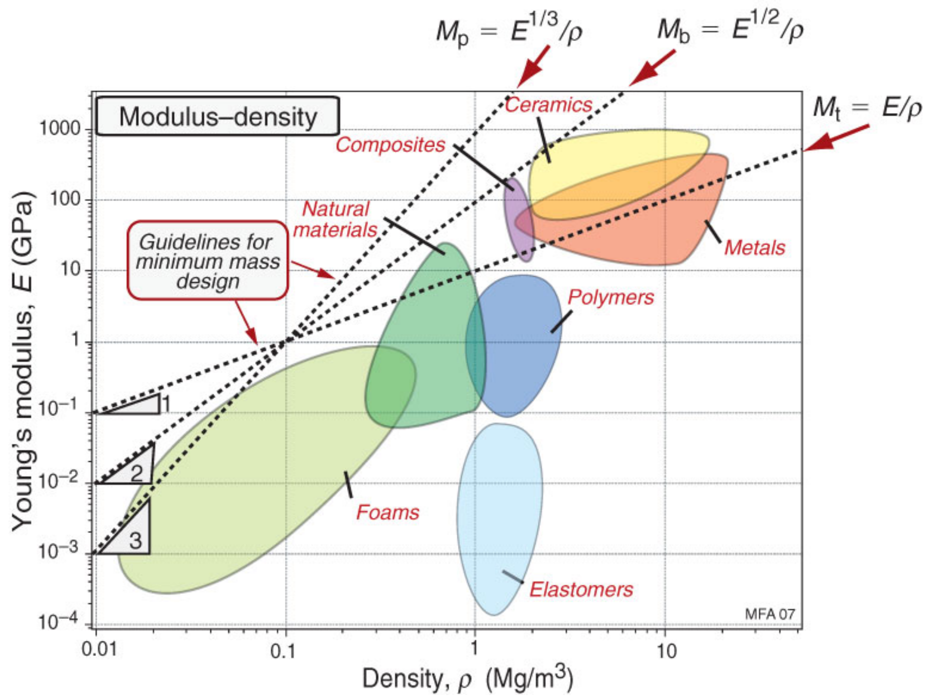


Figure 5.10 A schematic *E*–ρ chart showing guidelines for three material indices for stiff, lightweight structures.

- Family of parallel lines each one at a particular value of the material index of interest, M

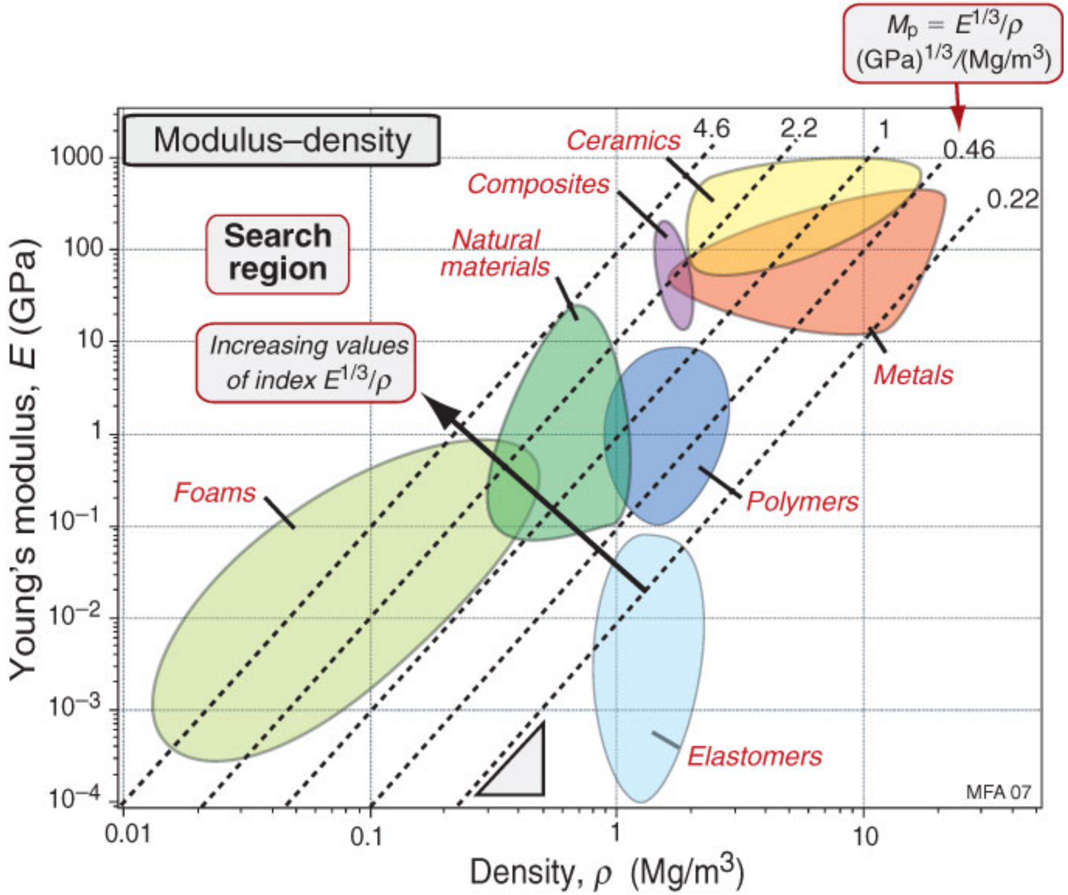


Figure 5.11 A schematic *E*–ρ chart showing a grid of lines for the index $E^{1/3}/\rho$.
The units are $(GPa)^{1/3}/(Mg/m^3)$.

Computer-aided selection

- For more complex problems with multiple constraints a computer-aided method is helpful (the CES software).

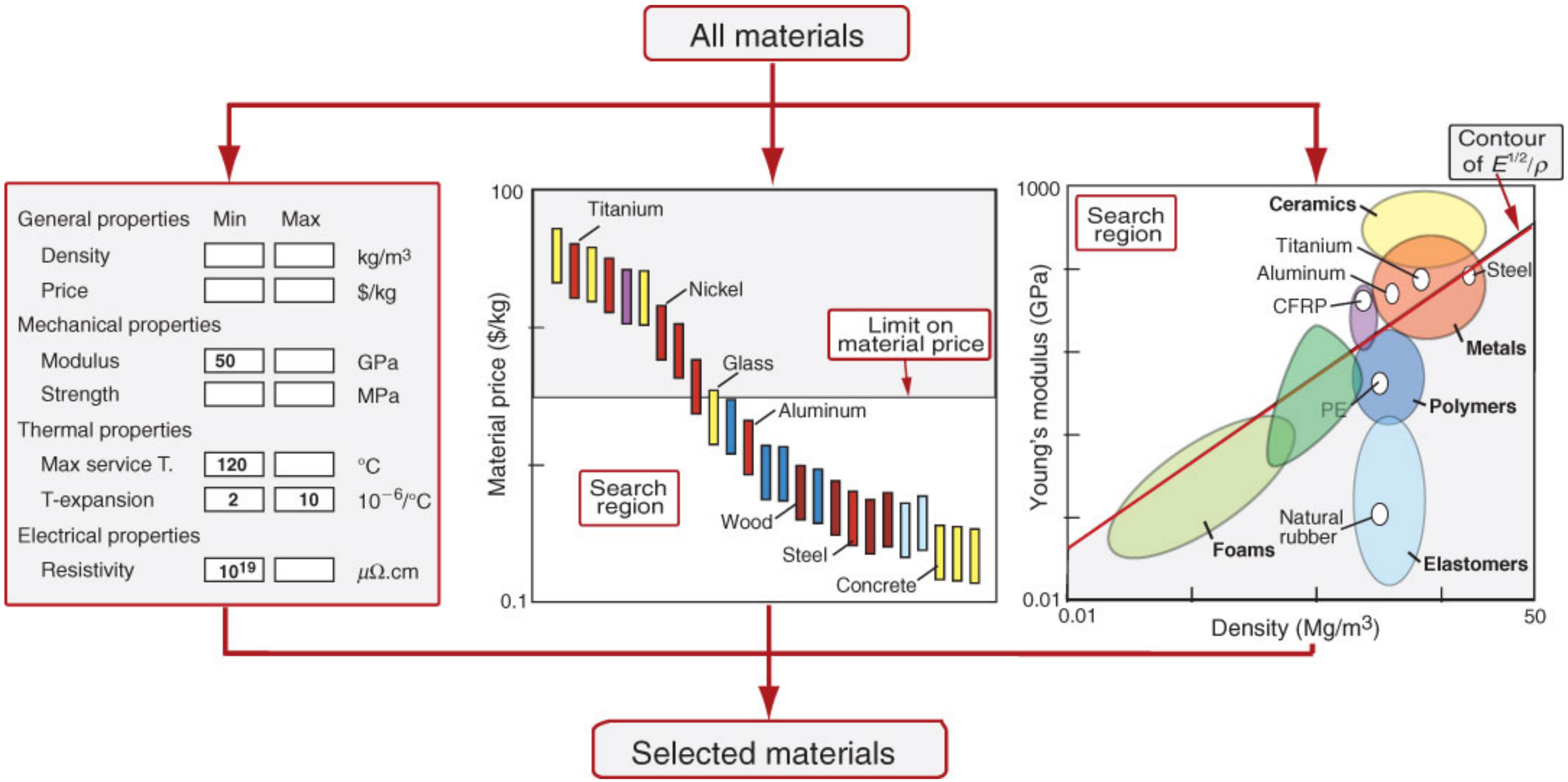


Figure 5.12 Computer-aided selection using the CES software. The schematic shows the three types of selection window. They can be used in any order and any combination. The selection engine isolates the subset of materials that pass all the selection stages.

5.5 Case studies

Light levers for corkscrews (light stiff beam)

- Constraints: length, L, rectangular section, maximum deflection δ, stiffness of S*, impact resistant
- Objective: minimize mass
- Free variables: material, area of cross section
- Loaded in bending: bending moment, M = FL
- Material index already derived $M_b = E^{1/2}/\rho$;

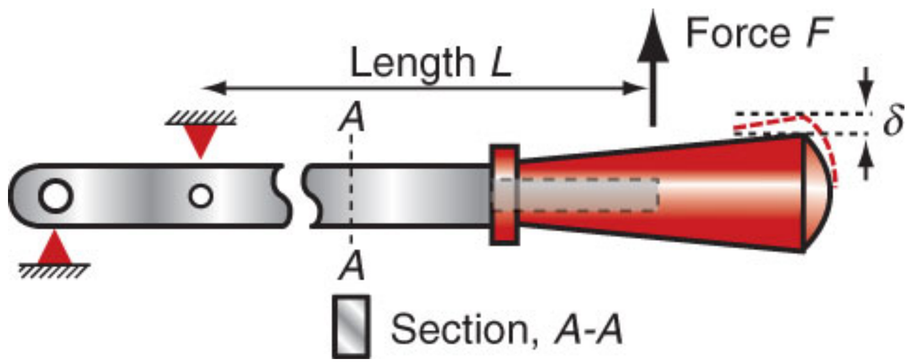


Figure 5.13 The corkscrew lever from Chapter 3. It must be adequately stiff and, for traveling, as light as possible.

- Selection line positioned to limit possibilities, some of which are too brittle.

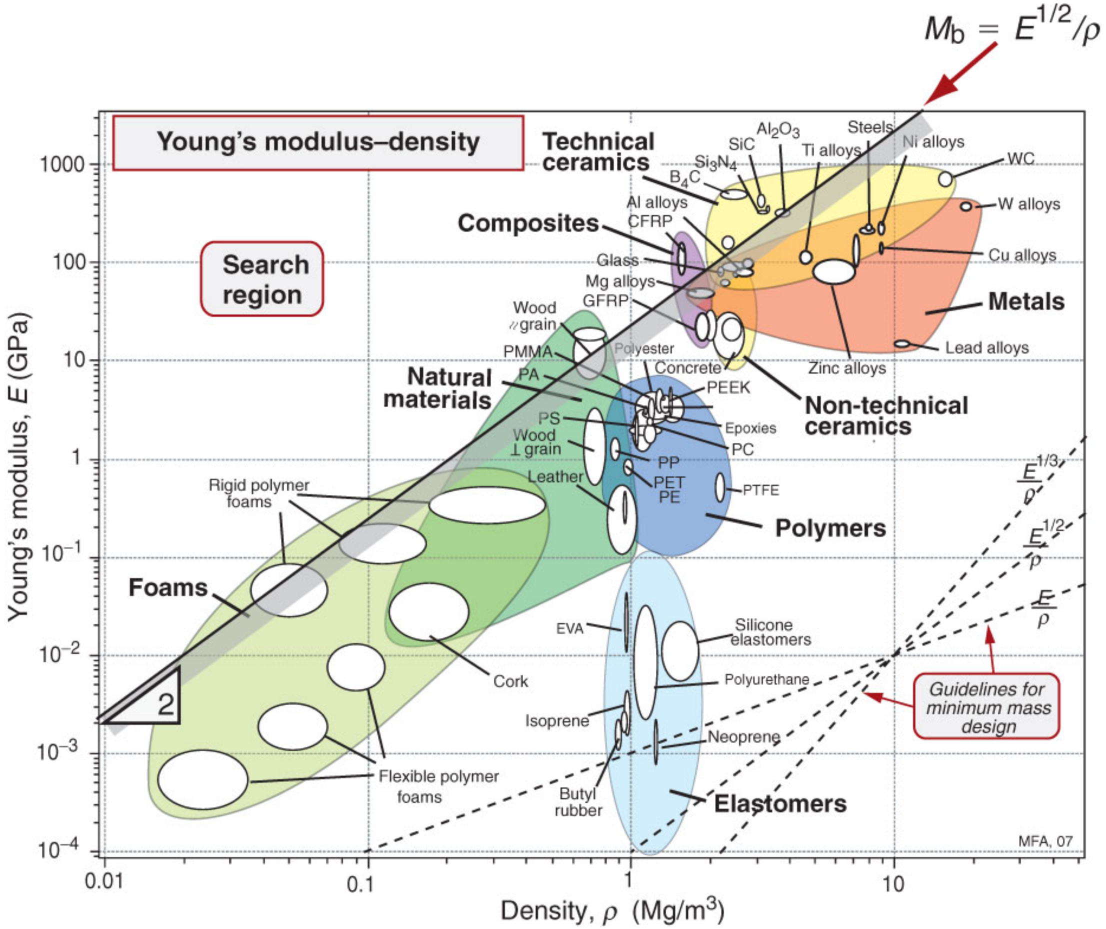


Figure 5.14 Selection of materials for the lever. The objective is to make it as light as possible while meeting a stiffness constraint.

Cost: structural materials for buildings (floor beam)

- Constraints: length, L, square section, maximum deflection δ, stiffness of S*
- Objective: minimize cost
- Free variables: material, area of cross section
- Material index already derived for a light stiff beam; adding cost: $C = mC_m = AL\rho C_m$
- Leading to material index $M = E^{1/2}/\rho C_m$

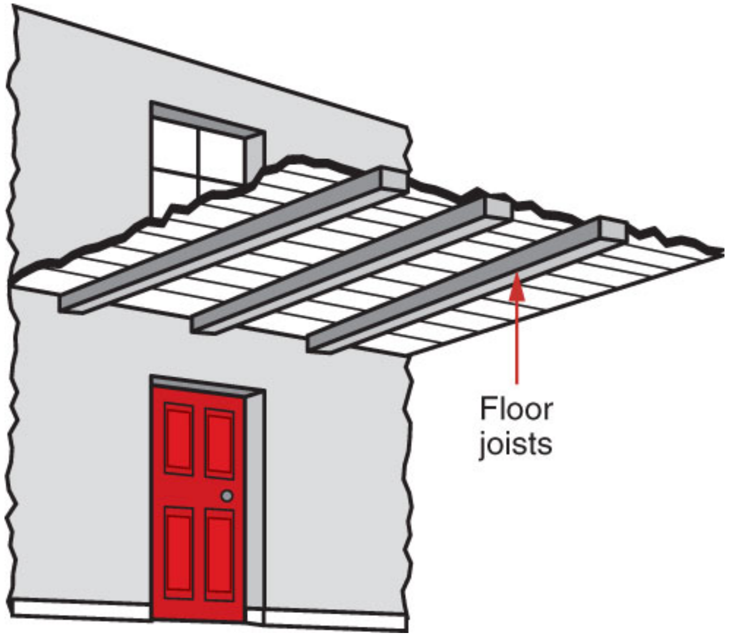


Figure 5.15 The materials of a building are chosen to perform three different roles. Those for the structure are chosen to carry loads. Those for the cladding provide protection from the environment. Those for the interior control heat, light and sound. Here we explore structural materials.

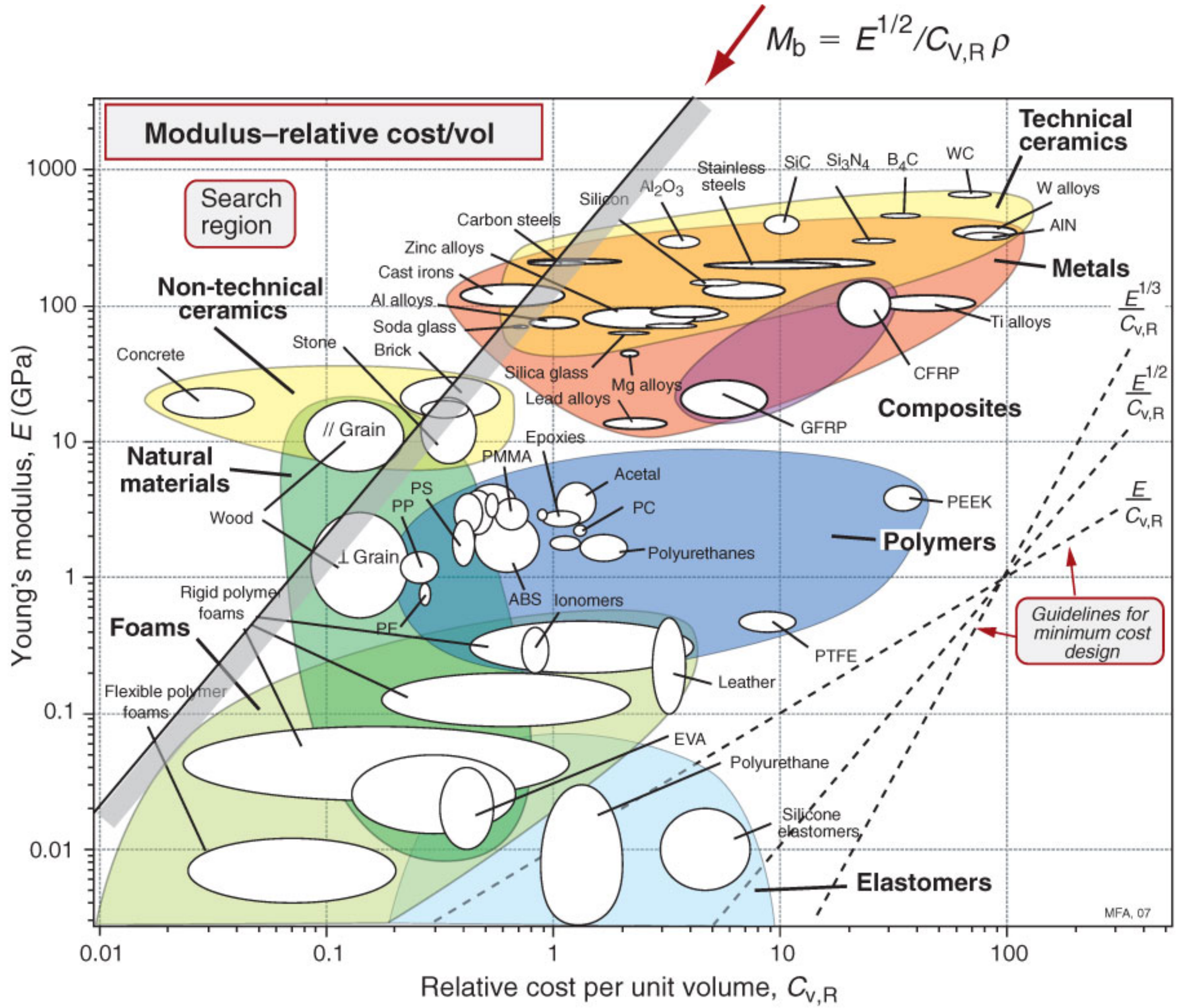
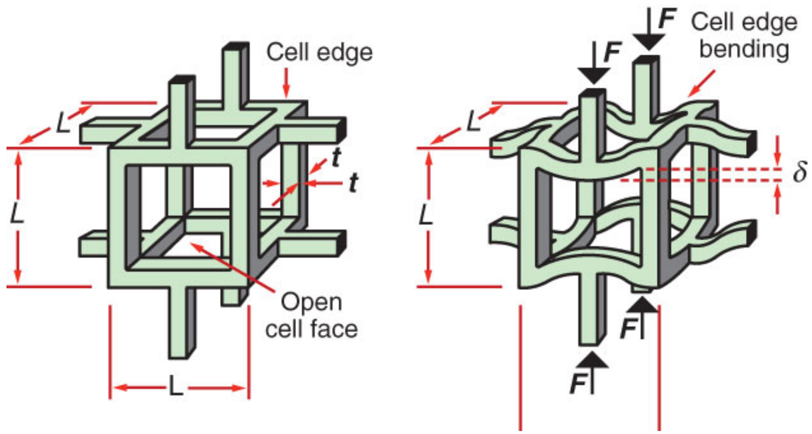


Figure 5.16 The selection of materials for stiff floor beams. The objective is to make them as cheap as possible while meeting a stiffness constraint. Concrete, stone, brick: only strong in compression. Wood, steel: good choices, also because they can be given efficient shapes

Cushions and padding: the modulus of foams

- Cellular solids are characterized by relative density $\rho^*/\rho_s = (t/L)^2$
- External stress σ, then $F = \sigma L^2$ and this force bends the cross beam at the center giving $\delta = FL^3/C_1E_sI$ {= Eq. 5.5, $I = t^4/12$}
- Compressive strain: $\varepsilon = 2\delta/L$ (see picture)
- Modulus of foam: $E^* = \sigma/\varepsilon$
- Combining everything: $E^*/E_s = (t/L)^4 = (\rho^*/\rho_s)^2$

Figure 4.23 Manipulating the modulus by making a foam—a lattice of material with cell edges that bend when the foam is loaded.

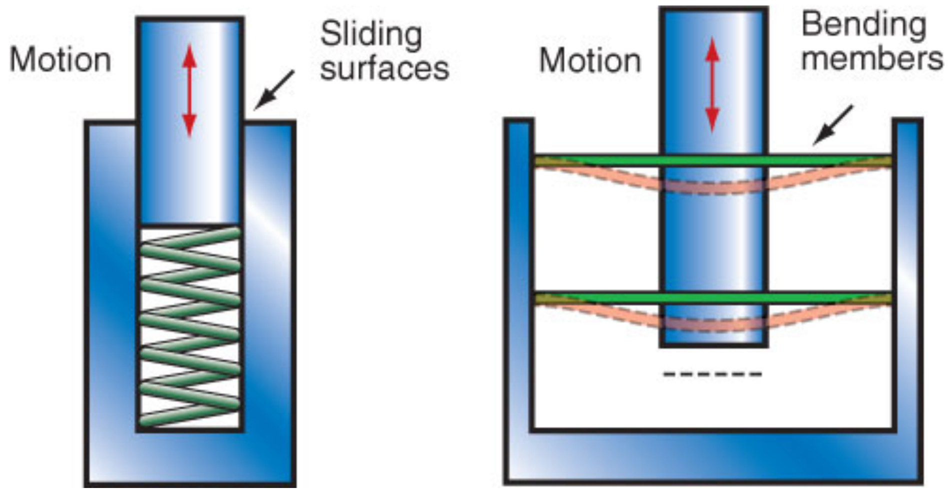


Vibration: avoiding resonance when changing material

- Natural frequencies are proportional to $(E/\rho)^{1/2}$
- For old, o, and new, n, materials: $\Delta f = (E_n\rho_o/E_o\rho_n)^{1/2}$

Bendy design: part-stiff, part-flexible structures

Figure 5.17 A sliding mechanism replaced by an elastic mechanism.



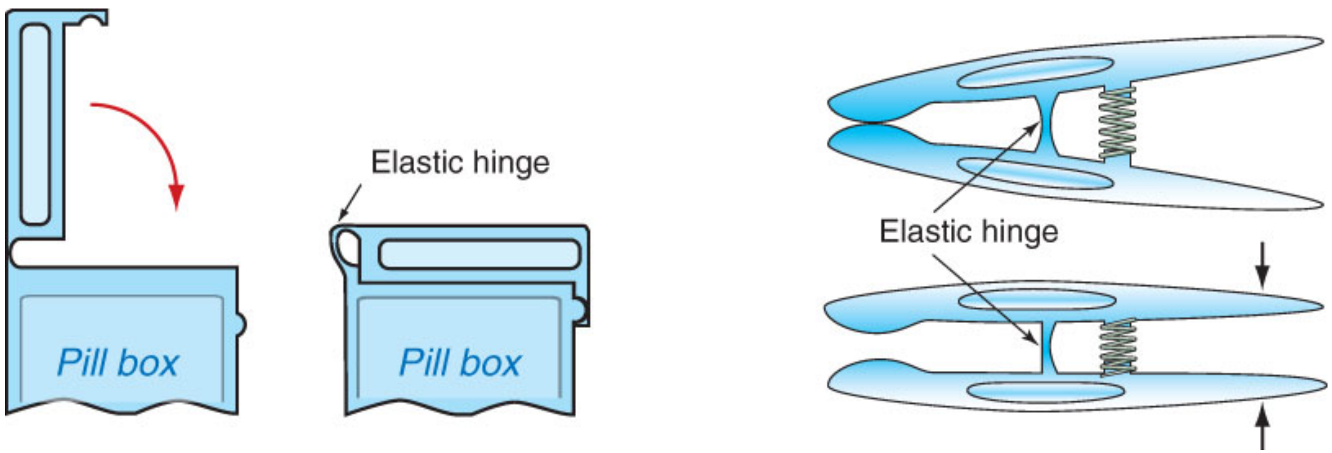


Figure 5.18 Elastic or 'natural' hinges allowing flexure with no sliding parts.

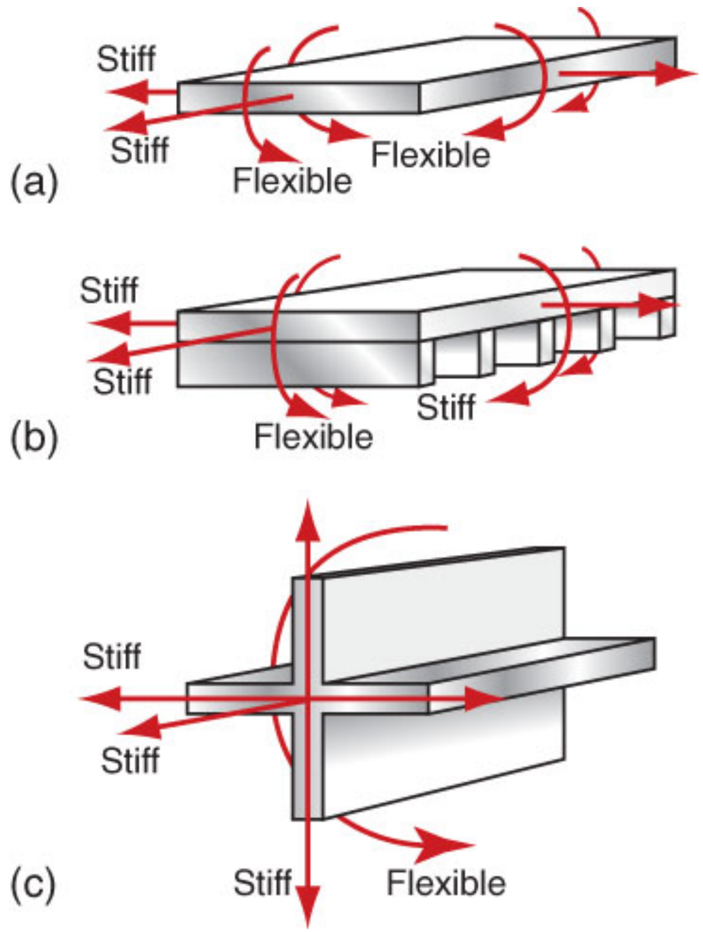


Figure 5.19 The flexural degrees of freedom of three alternative section shapes. (a) Thin plates are flexible about any axis in the plane of the plate, but are otherwise stiff. (b) Ribbed plates are flexible about one in-plane axis but not in others. (c) Cruciform beams are stiff in bending but can be twisted easily.