Chapter 7
Bend and crush: strength-limited design

Elastic design, avoiding plasticity, ensures that the cabin of the car does not deform in a crash.

Plasicity absorbs the energy of impact, and allows metals to be shaped and polymers to be molded

Hot rolling

7.1 Introduction and synopsis

- Stiffness-limited design avoids excessive elastic deformation.
- Strength-limited design avoids plastic collapse – generally means avoiding yielding. If the component is designed to remain elastic throughout it is termed elastic design (not always required – local yielding may be tolerable). Sometimes we want controlled plastic collapse (collision).

7.2 Standard solutions to plastic problems

- *Yielding of ties and columns*.

  The stress is uniform; if < σ_y the component remains elastic, otherwise it yields.

- *Yielding of beams and panels*.

  A bending moment generates a linear variation of longitudinal stress across the section: $\sigma/y = M/I = E\kappa$.

  For elastic deflection we focused on $E\kappa$. for yielding we look at M/I.

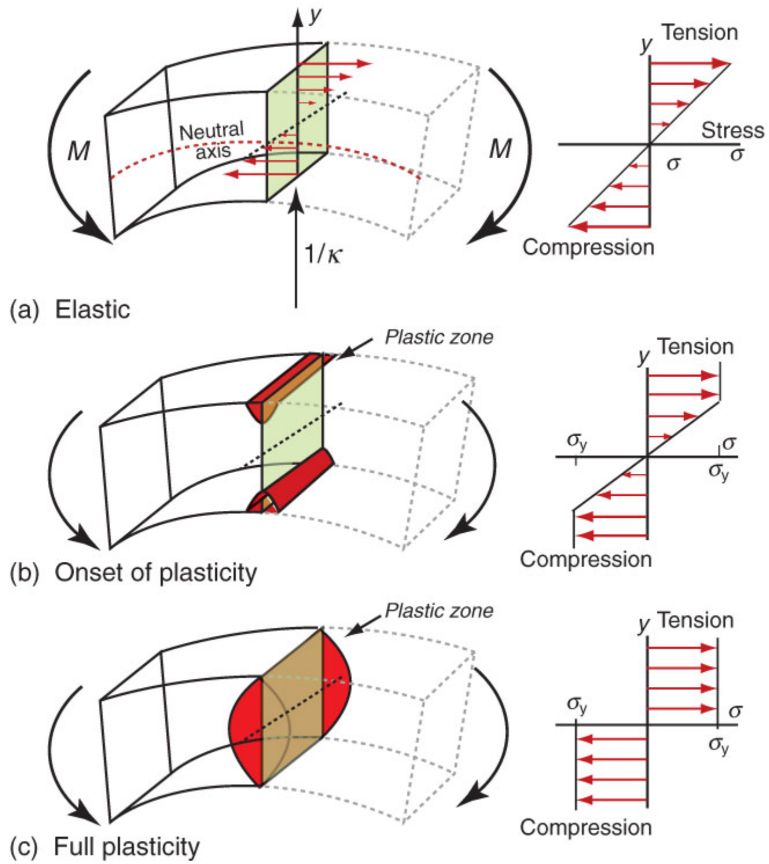


Figure 7.1 A beam loaded in bending. The stress state is shown for purely elastic loading (a), the onset of plasticity (b), and full plasticity (c).

7.2 Standard solutions to plastic problems

- $\sigma/y = M/I$, so the maximum stress occurs at the greatest distance from the neutral axis: $\sigma_{max} = My_{max}/I = M/Z_e$ (defines $Z_e = I/y_{max}$)

- Z_e is the *elastic section modulus*. Different from *elastic modulus (E)*

If $\sigma_{max} > \sigma_y$ small zones of plasticity appear where the stress is highest. Damage but so far not failure.

At higher moments the plastic zone penetrates the section and the stress profile is truncated. At some point the 'plastic hinge' closes, causing failure.

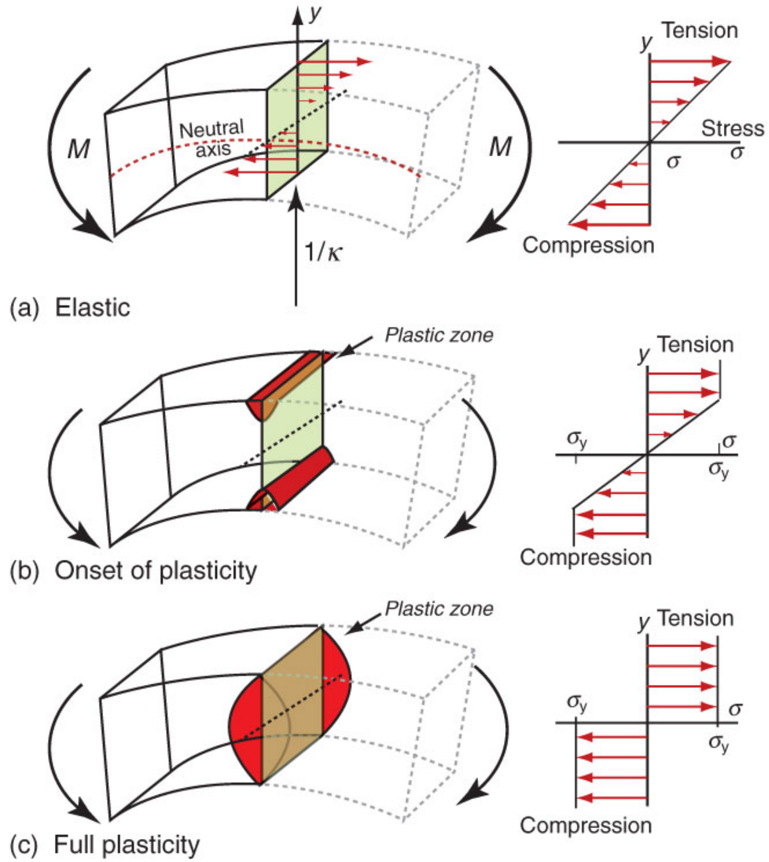


Figure 7.1 A beam loaded in bending. The stress state is shown for purely elastic loading (a), the onset of plasticity (b), and full plasticity (c).

- Three loading schemes with maximum moments FL, FL/4, and FL/8. “Plastic hinges” form at the red regions.
- The failure moment, M_f, is found by integrating the moment caused by the constant stress distribution (full plasticity) over the section.

d_y in book, Eq. (7.3)

- $M_f = \int_{section} b(y)y\sigma_y dy = Z_p\sigma_y$; Z_p is the *plastic section modulus*.
- Two new functions of section shape are defined for beam failure; Z_e for first yielding and Z_p for full plasticity. Z_p/Z_e is a measure of the safety margin. For a solid rectangle it is 1.5 while for tubes and I-beams it is closer to 1.

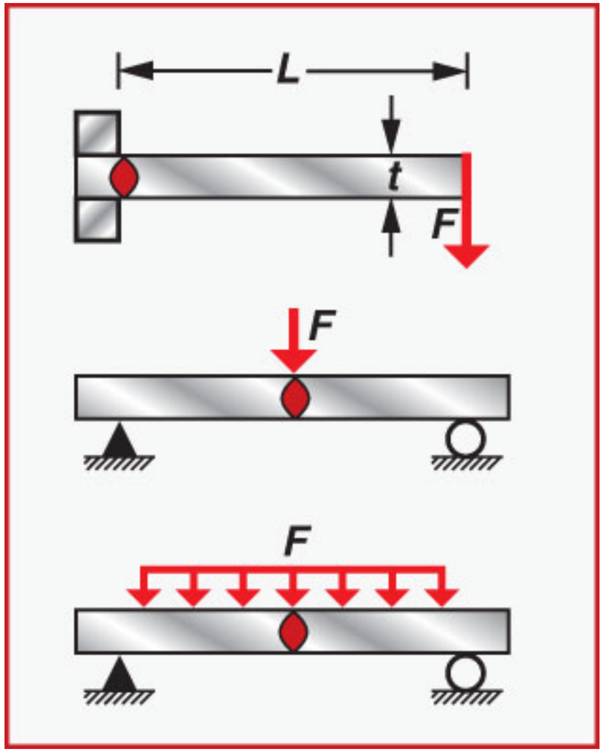
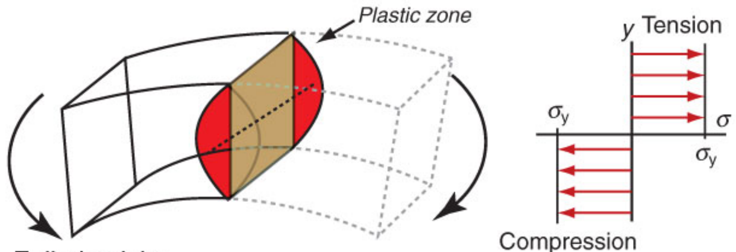


Figure 7.2 The plastic bending of beams.



First yielding

$Z_e = (1/y_m) \int_{section} b(y) y^2 dy$

Full pasticity

$Z_p = \int_{section} b(y) y dy$

Note: $h = 2y_m$

y

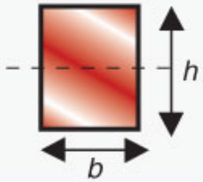
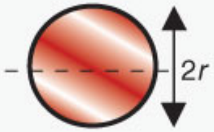
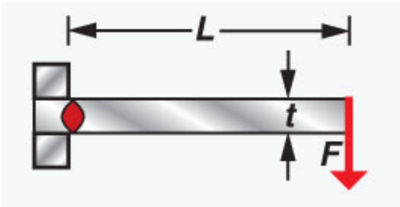
Section shape	Area A m^2	Elastic section modulus Z_e m^3	Plastic section modulus Z_p m^3
	bh	$\frac{bh^2}{6}$	$\frac{bh^2}{4}$
	πr^2	$\frac{\pi}{4} r^3$	$\frac{\pi}{3} r^3$
	$\pi (r_o^2 - r_i^2)$ $\approx 2\pi r t$	$\frac{\pi}{4 r_o} (r_o^4 - r_i^4)$ $\approx \pi r^2 t$	$\frac{\pi}{3} (r_o^3 - r_i^3)$ $\approx \pi r^2 t$

Figure 7.3 The area A, elastic section modulus Z_e and fully plastic modulus Z_p for three simple sections. In both cases the moment required is $M = Z\sigma_y$.

Example

means: stress at y_m



L = 250 mm
t = 1 mm
F = 10 N
Moment M = FL

Example 7.1

Calculate the maximum ‘extreme fibre’ stress in the 300 × 25 × 1 mm stainless steel ruler in Examples 5.1 and 5.2, when it is mounted as a cantilever with the X-X axis horizontal with 250 mm protruding and a weight of 10 N is hung from its end. Compare this bending stress with the yield stress of the material and the direct stress when a 10 N force is used to pull the ruler axially.

- *Yielding of shafts*. Recall $\tau = Tr/K = G\theta r/K$; K is the polar second moment of inertia. Failure occurs when the max surface stress > σ_y.
- The max shear stress τ_{max} is at the surface: $\tau_{max} = TR/K$ where R is the shaft radius. Ch. 6: The yield stress in shear, k, is half the tensile yield stress so first yield occurs at $\tau_{max} = \sigma_y/2$ (or $\sigma_y/3$) . The maximum torque occurs when τ = k over the whole section.
- For a solid circular section the collapse torque is $T = 2\pi R^3 k/3$.

Eq. 7.6, p.146, incorrectly has *r* instead of *R*

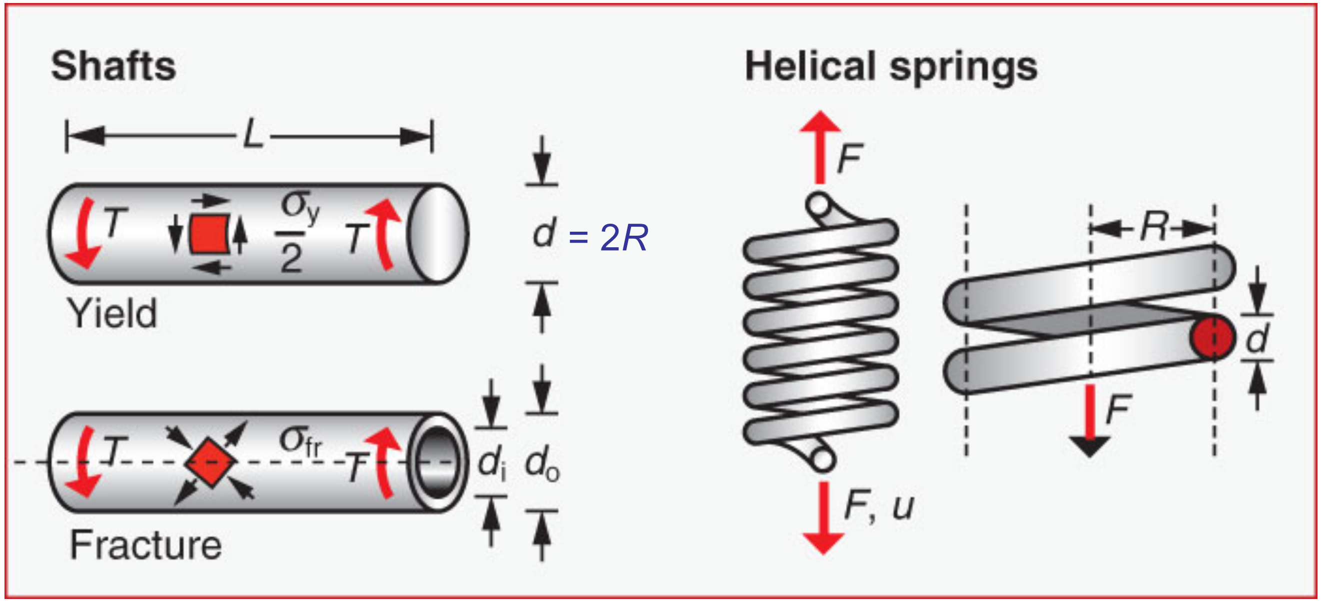


Figure 7.4 Elastic torsion of shafts. The stress in the shaft depends on the torque *T* and the polar moment of area *K*. Helical springs are a special case of torsional loading.

- When a *helical spring* is loaded axially the *turns* are loaded torsionally.

The stiffness is given by $S = F/u = Gd^4/64nR^3$ (G = shear modulus, n = number of turns). Elastic extension is limited by onset of plasticity at $F_{crit} = \pi d^3 \sigma_y / 32R$.

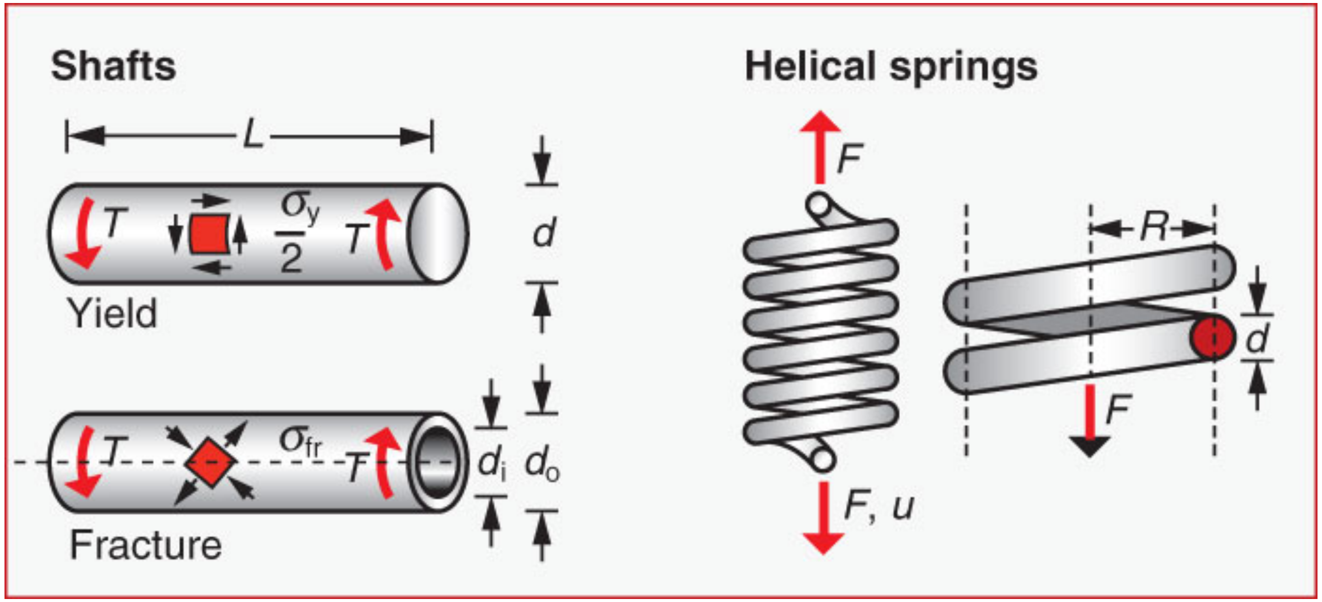


Figure 7.4 Elastic torsion of shafts. The stress in the shaft depends on the torque *T* and the polar moment of area *K*. Helical springs are a special case of torsional loading.

Example

Example 7.2

The solid brass shaft in Example 5.2 has a yield strength in tension of 300 MPa. Calculate the torque at first yield and the torque at plastic collapse (that is, it is plastic right through).

- *Spinning discs (flywheels)*. Centrifugal forces generate a radial tensile stress that reaches a max of σ_{max}.
- For a Poisson's ratio of 1/3 the kinetic energy turns out to be $U = \pi\rho t\omega^2R^4/4$ and the maximum stress $\sigma_{max} = 0.42\rho\omega^2R^2$. ρ is the density, ω is the angular velocity.
- The disc yields when $\sigma_{max} = \sigma_y$.

Note: "centrifugal forces" as a result of rotation do not really exist, centipetal force do

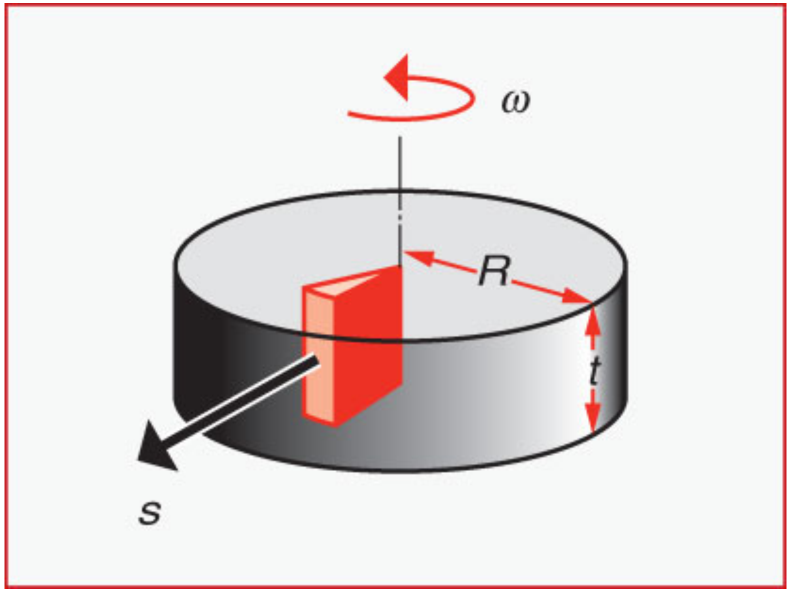


Figure 7.5 Spinning disks, as in flywheels and gyroscopes, carry radial tensile stress caused by centrifugal force.

Question
Would another material than CFRP be better?

“at the centre”?

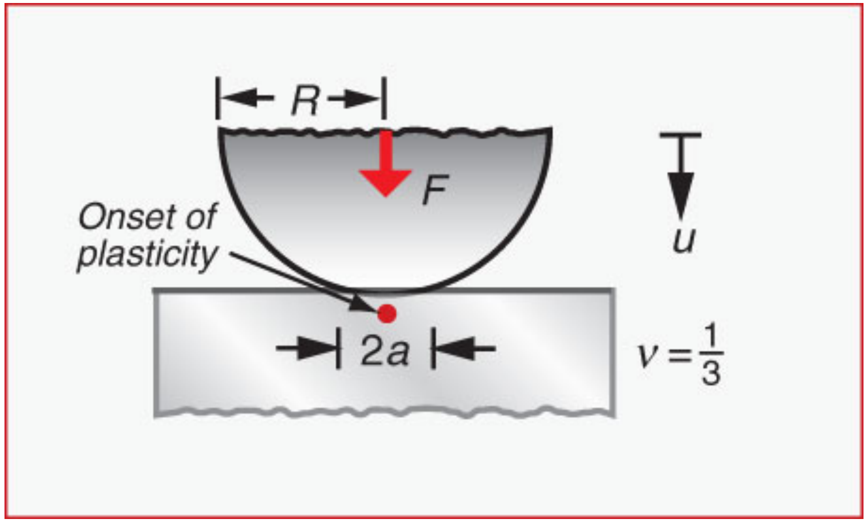
Example

Example 7.3

A flywheel has a radius of 150 mm and a thickness of 50 mm. It is made of CFRP with a strength of 700 MPa and a density of 1600 kg/m^3. What is its burst speed (the speed at which the material at the centre reaches its failure strength)? What is the maximum amount of kinetic energy it can store? How does this value compare with the chemical energy stored in the same mass of gasoline?

- *Contact stresses*. Yielding at contacts is closely linked to failure by wear and fatigue. Loaded contact points flatten elastically and the contact area grows. First analyzed by Hertz.
- For a sphere of radius R and modulus E pressed against a flat surface with a load F the radius of contact in the elastic regime is: $a \approx 0.7(FR/E)^{1/3}$ for a Poisson ration of 1/3.
- The relative displacement of the two bodies is $u \approx -(F^2/E^2R)^{1/3}$.
- For failure we consider the max value of the shear stress which occurs at a depth of $\approx a/2$. $\tau_{max} = F/2\pi a^2$ and a plastic zone appears if this value exceeds the shear yield strength $\approx \sigma_y/2$ (or $\sigma_y/3$) .

Figure 7.6 Contact stresses are another form of stress concentration. When elastic, the stresses and displacement of the surfaces towards each other can be calculated.



Example

Example 7.4

A steel ball bearing of radius 5 mm is pressed with a force of 10 N onto a steel ball-race. Both bearing and race have a Young's modulus of 210 GPa and yield strength of 400 MPa. Calculate the contact radius and the relative displacement of the two bodies. Compare the maximum stress beneath the contact patch with the strength of the surface and with the stress when the load of 10 N is uniformly distributed over a circular section radius 5 mm.

- *Stress concentrations*. Holes, slots, threads, and changes in section concentrate stress locally. Yielding starts here but initially not catastrophic (the effect on fatigue is much more threatening)
- We define the *nominal* stress in a component σ_{nom} as the load divided by the *smallest* cross-section, ignoring stress raising features.
- The *maximum local stress* is given by multiplying the nominal by a *stress concentration factor* K_{sc}: $K_{sc} = \sigma_{max}/\sigma_{nom} = 1 + \alpha(c/\rho_{sc})^{1/2}$ where ρ_{sc} is the minimum value of the radius of curvature of the feature, c is a characteristic dimension of the feature, and α is ≈2 for tension but ≈ ½ for torsion and bending.

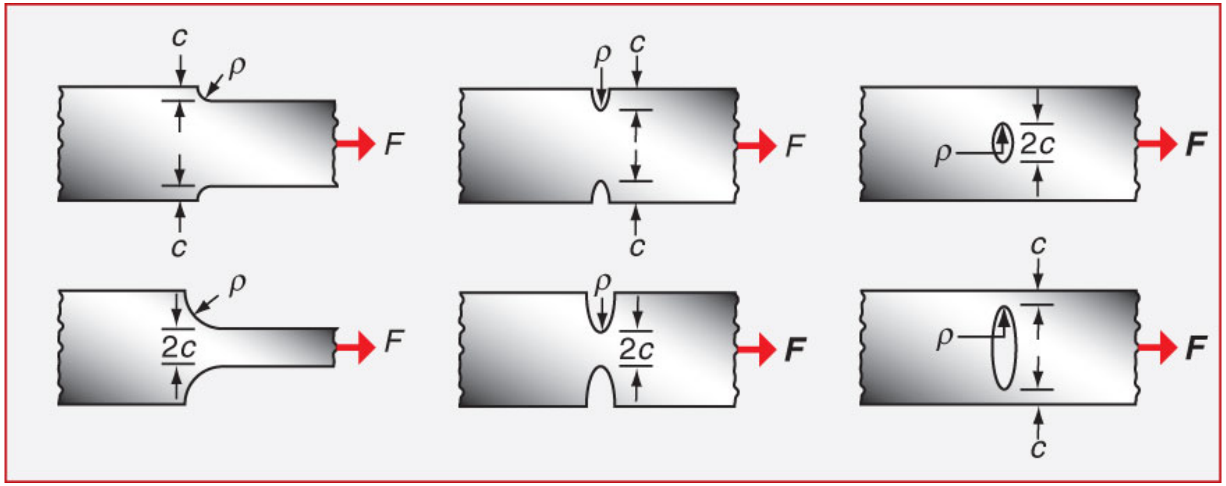


Figure 7.7 Stress concentrations. The change of section concentrates stress most strongly where the curvature of the surface is greatest.

Example

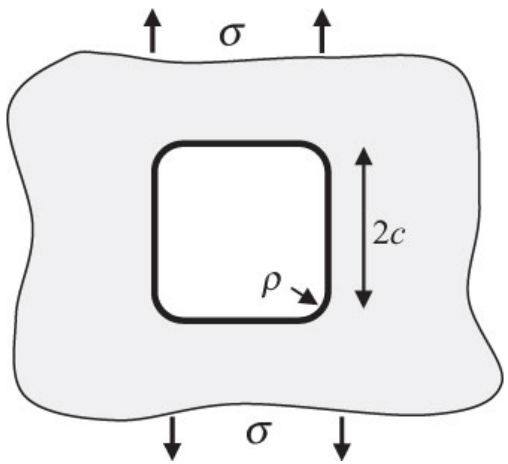
Example 7.5

fuselage = **romp**

The skin of an aircraft is made of aluminum with a yield strength of 300 MPa and a thickness of 8 mm. The fuselage is a circular cylinder of radius 2 m, with an internal pressure that is $\Delta p = 0.5$ bar (5×10^4 N/m^2) greater than the outside pressure. A window in the side of the aircraft is nominally square (160 mm × 160 mm). It has corners with a radius of curvature of $\rho = 5$ mm.

hoop = **ring, hoepel, cirkel**

Calculate the nominal hoop stress in the aircraft wall due to pressure and estimate the peak stress near the corner of the window. Ignore the effects of longitudinal stress in the tube. Note that the hoop stress in the wall of a circular pressure vessel of radius R and thickness t with internal pressure Δp is given by $\sigma = \Delta pR/t$. If the maximum allowable stress is 100 MPa, is 5 mm a suitable radius for the corners of the window?



De Havilland Comet 1, 1951

De Havilland Comet 2, 1954

7.3 Material indices for yield-limited design

- *Minimizing weight: a light, strong tie-rod*. Constraint: no yielding in tension, length L specified. Objective: minimize the mass m. Free variables: cross-section A, material.
- $M = AL\rho$ and $F/A \le \sigma_y$
- Eliminating A gives $m \ge FL(\rho/\sigma_y)$; inverting we seek materials with largest values of $M_t = \sigma_y/\rho$ which is the *specific strength*.

  Ti alloys and CFRP excel.

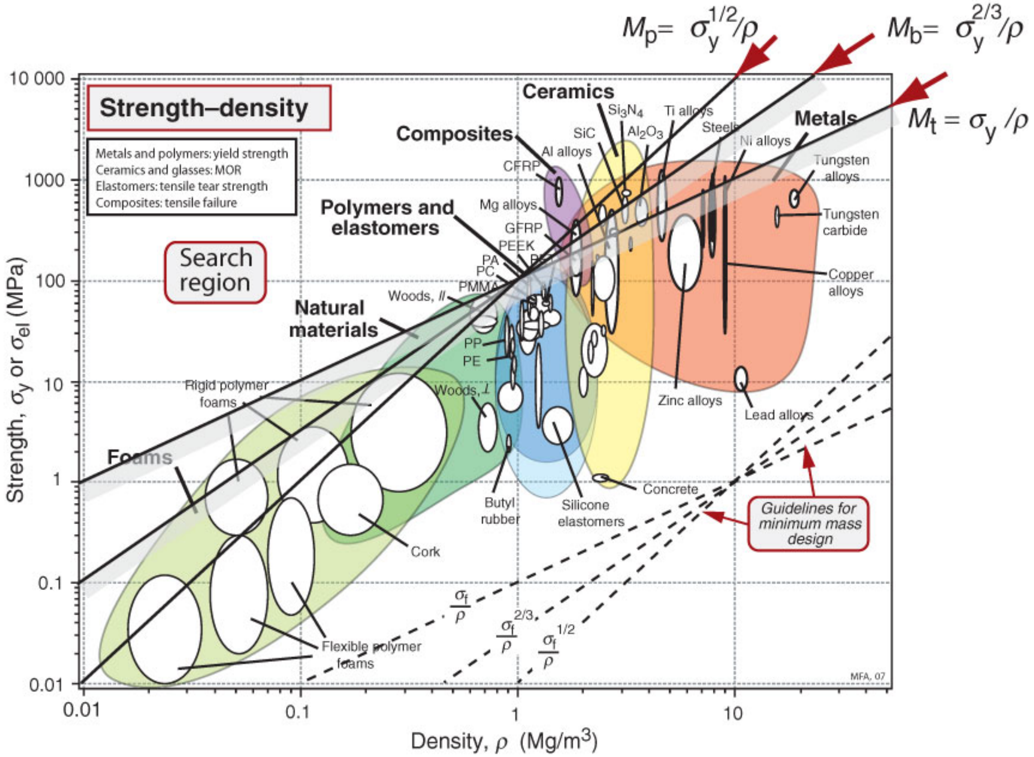
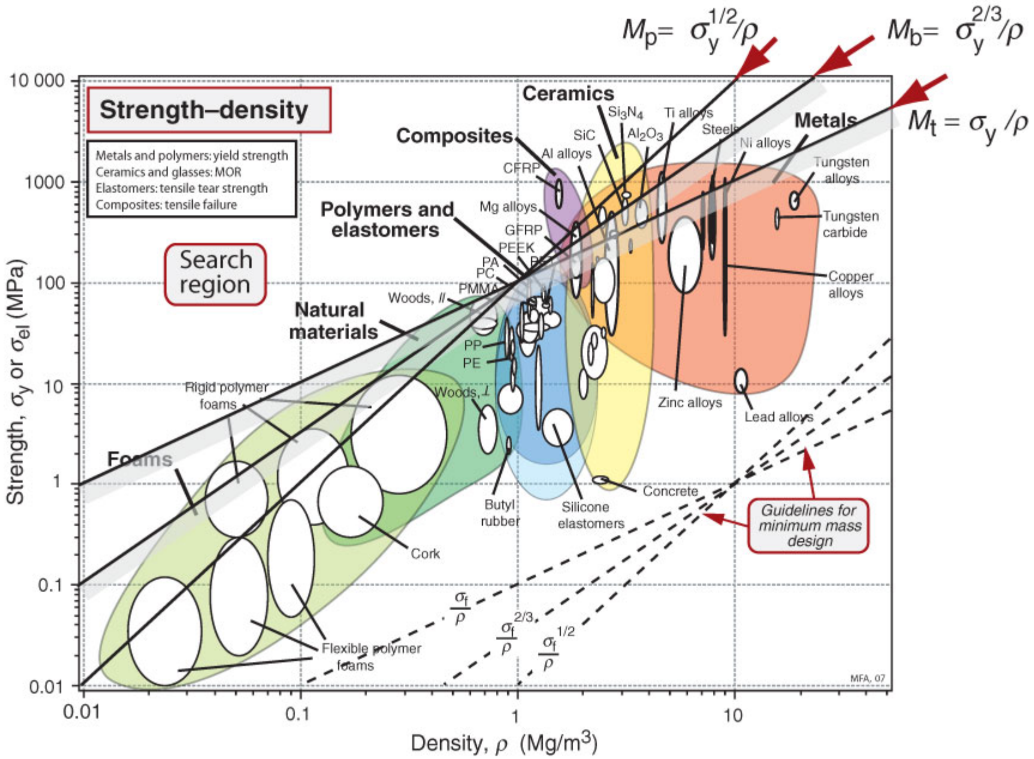


Figure 7.8 The strength–density chart with the indices σ_y/ρ, $\sigma^{2/3}_y/\rho$ and $\sigma_y^{1/2}/\rho$ plotted.

- *Minimizing weight: light, strong panels*. Constraint: no yielding in bending, width b and span L specified. Objective: minimize the mass m. Free variables: thickness h, material.
- Following standard procedure $M_p = \sigma_y^{1/2}/\rho$. Mg, Al, Ti alloys, GFRP, wood, all outperform steel and CFRP excels.



- *Light strong beams: the effect of shape*. Constraint: no yielding in bending, span L square section specified. Objective: minimize the mass m. Free variables: cross-section A, material.
- Standard procedure: $M_b = \sigma_y^{2/3}/\rho$. Holds for self similar shapes.
- We can gain strength by increasing I through shape change (tube, I-beam) but this is material dependent. The shape factor for strength: $\phi_B^y = Z_e^{shaped}/Z_e^{solid}$.

Note: Increasing stiffness ($\propto$I) by shaping is more effective than increasing strength ($\propto Z_e = I/y_m$) by shaping. This is because increasing I often also increases y_m.

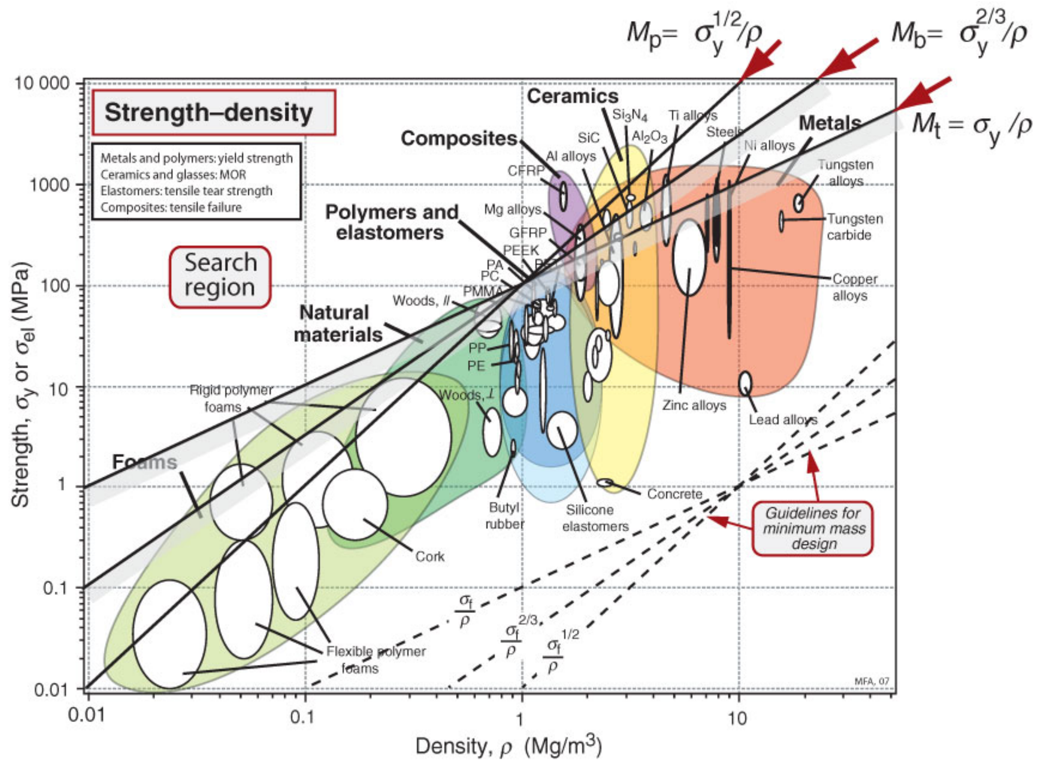
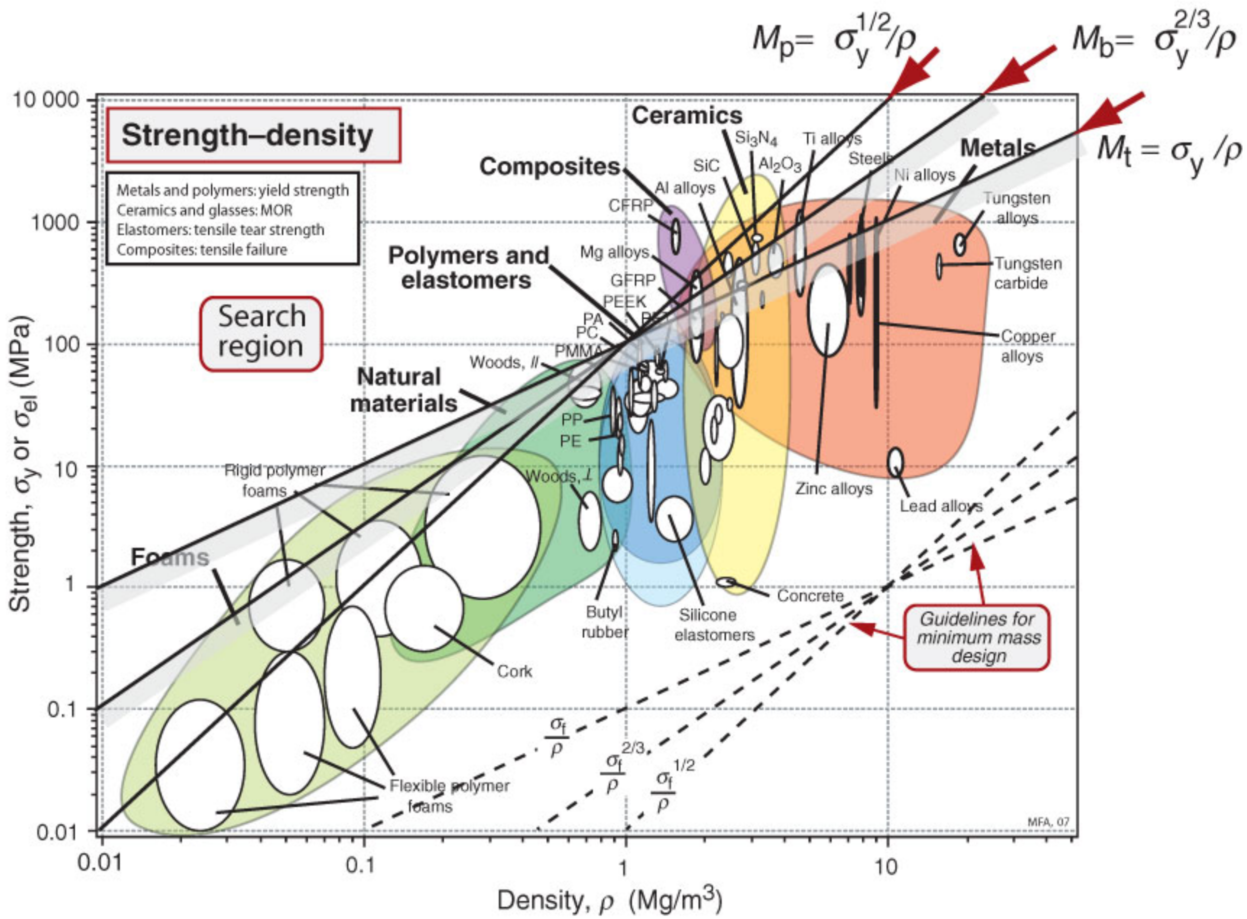


Table 7.4 The effect of shaping on strength and mass of beams in different structural materials

Material	Maximum failure shape factor ϕ_B^y (failure moment relative to solid beam)	Mass ratio by shaping (relative to solid square square beam)
Steels	13	0.18
Al alloys	10	0.22
Composites	9	0.23
Wood	3	0.48

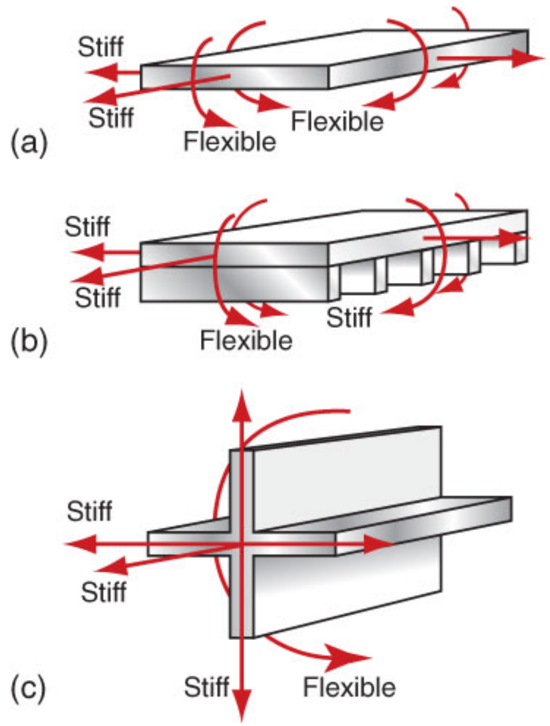
- *Minimizing material cost or volume*. $C = mC_m = ALC_m\rho$ leading to the same indices as before with ρ replaced by $C_m\rho$.

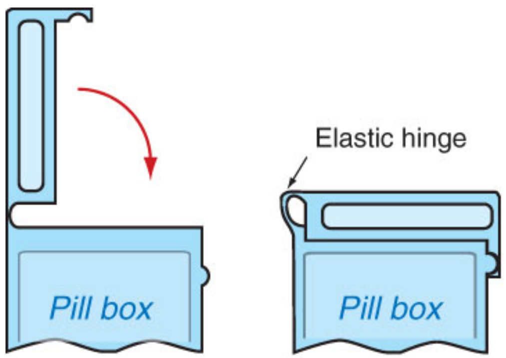
- *Corkscrew levers again: strength*. Following standard procedure $M_p = \sigma_y^{2/3}/\rho$. The selection is almost the same as for stiffness: CFRP, Mg and Al alloys.

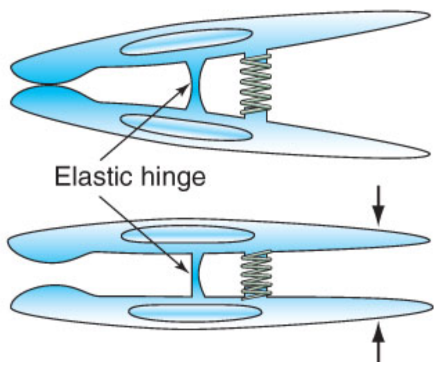


7.4 Case studies

- *Elastic hinges and couplings*. Consider the hinge for the lid of a box – one piece molding, no pins, screws, etc. A thin ligament (**band**) that flexes elastically but carries no significant axial loads. Seek a material that bends to the tightest radius without yielding or failing.
- A ligament of thickness t bent to radius R: the surface strain is ε = (t/2)/R. (Inside surface compressive strain is –(t/2)/R.)
- The max stress is σ = Et/2R which must not exceed σ_y.
- R ≥ (t/2)[E/σ_y]; therefore index M = σ_y/E
- Polymers are best (generally PE, PP, nylon)







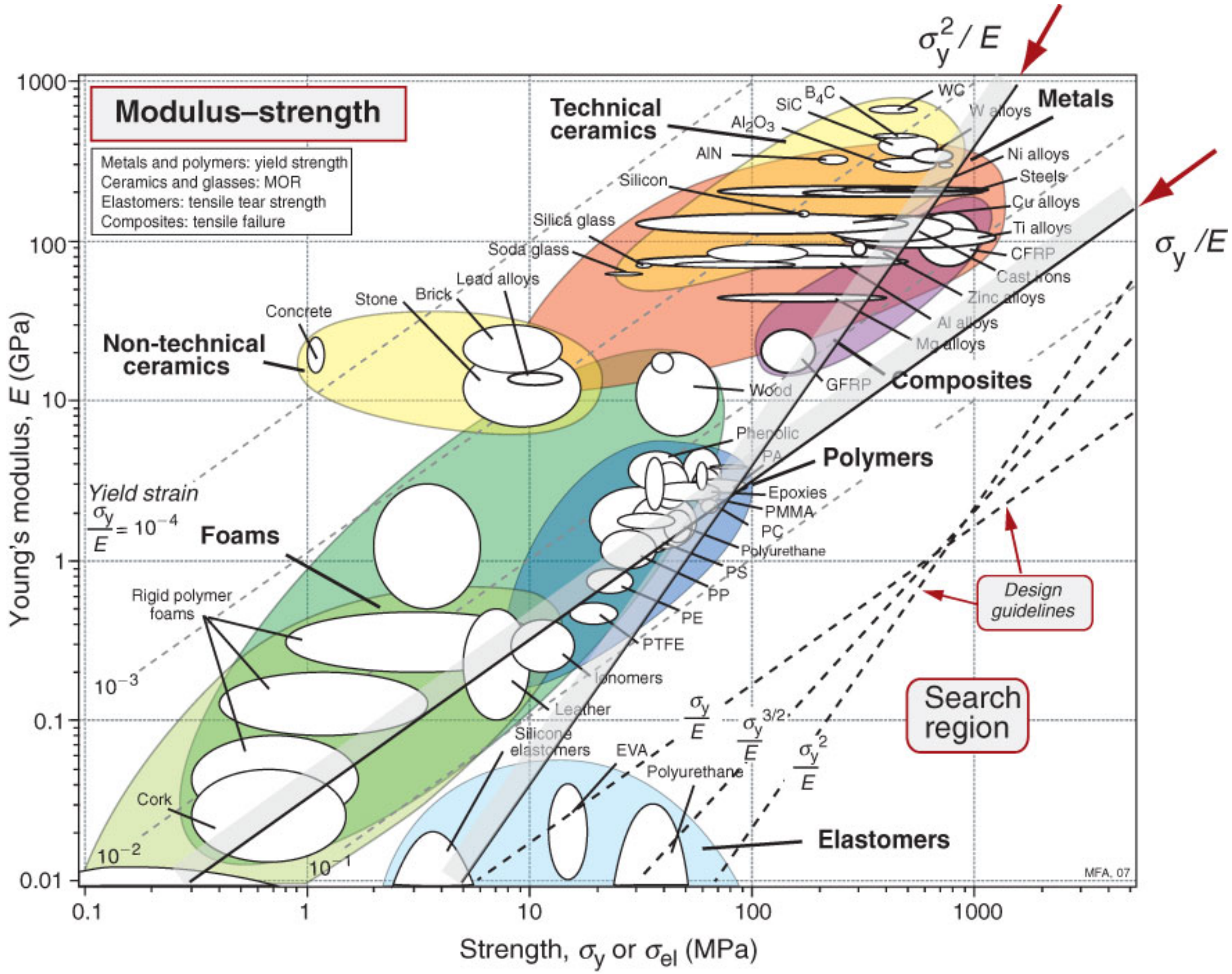


Figure 7.9 Materials for elastic hinges and springs. Polymers are the best choice for the former. High-strength steel, CFRP, and certain polymers and elastomers are the best choice for the latter.

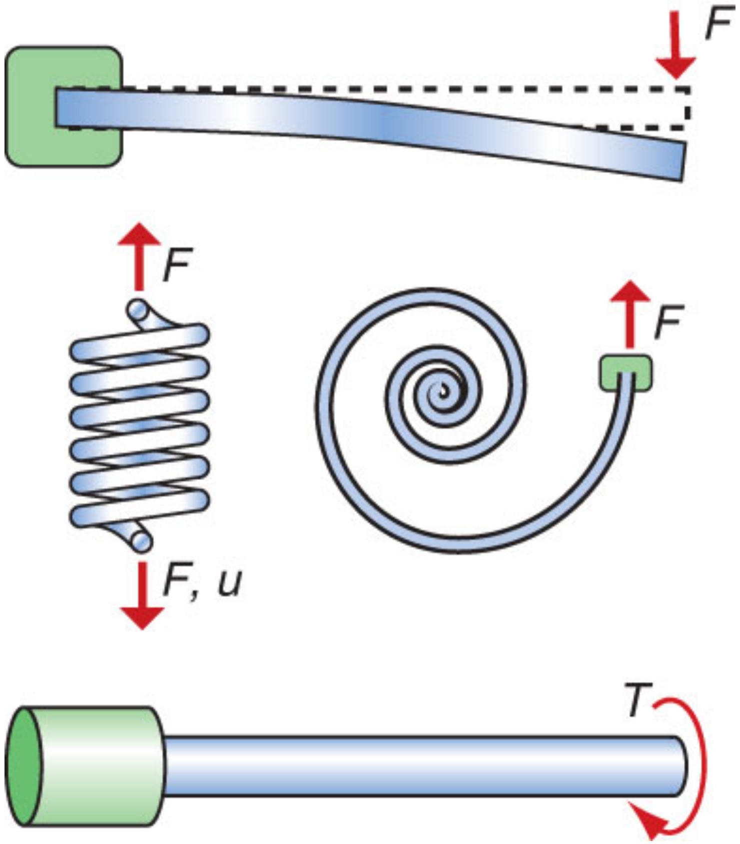


Figure 7.10 Springs: leaf, helical, spiral and torsion bar. Springs store energy. The best material for a spring, regardless of its shape or the way it is loaded, is one with a large value of σ^2_{el}/E, as we will see

- *Materials for springs*. Maximum σ must not exceed σ_y when the stored energy is $\sigma_y^2/2E$ per unit volume. Constraint: no failure. Objectives: max stored energy per volume. Free variables: material.
- M = σ_y^2/E; possibilities: high-strength steel, CFRP, titanium alloys, nylon, elastomers.

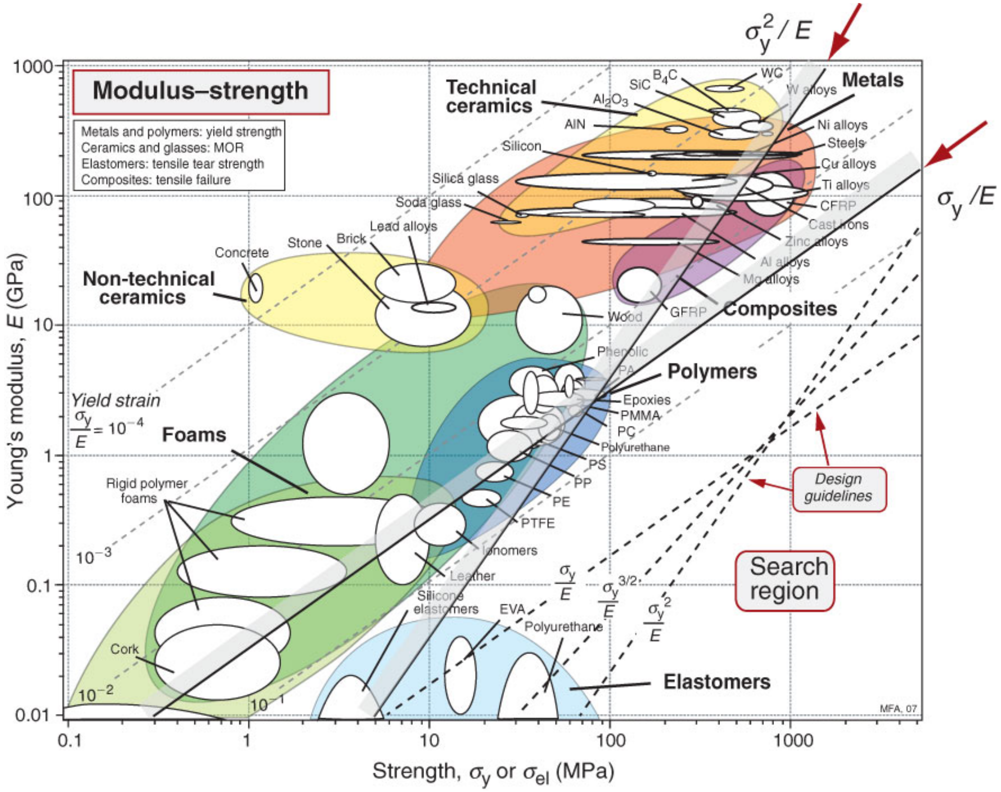
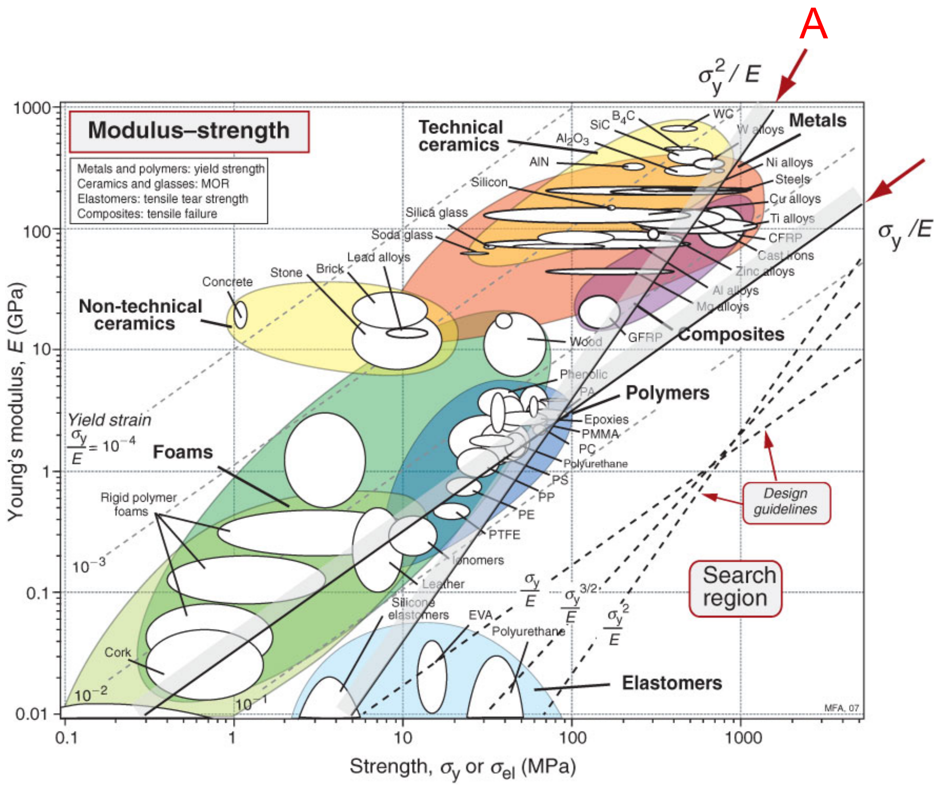


Figure 7.9 Materials for elastic hinges (polymers are the best choice) and springs (high-strength steel, CFRP, and certain polymers and elastomers are the best choice).

PI question

The best (stored energy per unit volume) materials for springs are those with maximum value of σ_y^2/E. Look at line A in the diagram. Which of these statements is false?

1. Silicone elastomers and high-strength steels are equally good
2. Materials under the line are better than those above it
3. All materials on the line show about equal deflection when equally loaded



- *Full plasticity: metal rolling*. A lower bound for the torque and power required for rolling is found from the plastic work, $\sigma_y \varepsilon_{pl}$ per unit volume required to produce a plastic strain ε_{pl} of $\Delta t/t_0$ ($\Delta t = t_0 - t_1$).
- If rolls rotate $\Delta\theta$ a length $R\Delta\theta$ and thus volume $V = R\Delta\theta t_0$ per unit width is fed into the bite where it is compressed to t_1.
- Equating the work done by a pair of rolls, $2T\Delta\theta$, to the plastic work, $V\sigma_y\varepsilon_{pl}$, gives the torque per roll: $T = R\sigma_y\Delta t/2$
- The power is the torque times the angular velocity ω radians per second:

  $P = 2T\omega = R\omega\sigma_y\Delta t$. Hot rolling takes less power (because σ_y is smaller).
- These are lower bounds; friction, sliding, and work hardening all increase the torque and power that is needed.

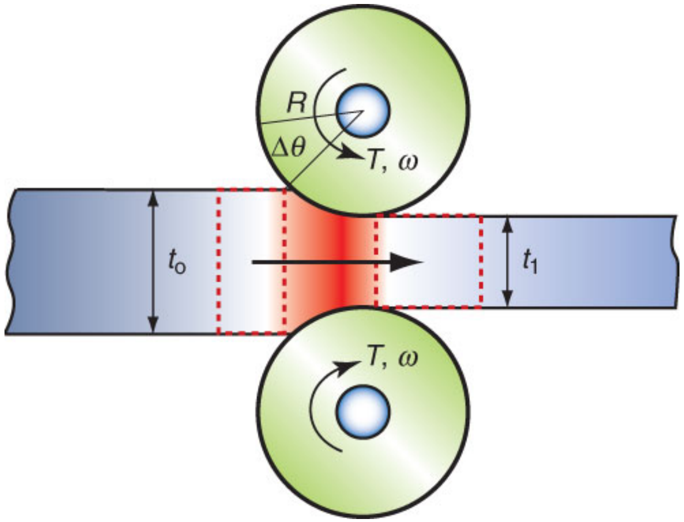


Figure 7.11 Rolling.

Example

Example 7.6

A rolling mill has a pair of rolls of radius 100 mm that rotate at an angular velocity of 0.5 rad/s. A steel strip passes through the rolls and its thickness is reduced from 25 mm to 18 mm. (a) Calculate the nominal plastic strain in the strip. (b) Calculate the power required to drive the rolls if: (i) the steel is at room temperature and (ii) if the steel is at 600 °C. The variation of the yield strength of the steel with temperature is shown below.