Exercises Mesoscopic physics

Exercises indicated with an * are more challenging and require a background in mathematics and quantum mechanics at the level of the bachelor applied physics.

Lecture 1: Introduction, Level Spacing and Density of States

1a When was the field of mesoscopic physics born? (which decade)
b Which development made the field possible?

2a Which fundamental equation is used to calculate the level spacing?
b What are the eigenvalues of an one-dimensional harmonic oscillator with $V(x) = ½m\omega^2x^2$?
c Calculate the density of states for a small frequency ω.
d Compare the energy dependence of the d.o.s. with that of a particle in an one-dimensional box.
e What would you take as the size L of the system when it is filled with 1000 non-interacting electrons?

3a Calculate the Fermi wavenumber k_F for a two dimensional electron gas (2DEG), expressed in the electron concentration n. Do the same for 1D.
The units in which the electron "concentration" is expressed changes with the dimensionality of the system. To compare systems with different dimensionalities, the concentration can be expressed in the effective distance between electrons $d_{e\text{-}e}$ by assuming that each electron lives in a box of size $d_{e\text{-}e}$.
b Express the results for k_F in one, two and three dimensions in $d_{e\text{-}e}$ and calculate the numerical value of the prefactors. Are you surprised that they are close to one?

4 Graphene is recently discovered material, consisting of an one carbon atom thick layer. The atoms are positioned in a hexagonal lattice with a distance d_{CC} = 1.4 Å between the atoms.
a Each carbon atom has one free electron. Find the value for the Fermi energy and Fermi wavelength using the free electron model.
b The band structure can also be calculated with a tight-binding model. Look up what the band structure $E(k)$ is in this case. What is the effective mass?
c* Find the density of states from the band structure. Compare the result to the results for a free electron gas.

5a Argue in which case the infinite square well describes reality the best: a metallic or a semiconducting material.
b Calculate the level spacing for a cubic gold cluster with size 1×1×1 and 10×10×10 nm^3 and compare this with its charging energy.
c Calculate the values for E_F (in eV), k_F (in nm^{-1}) and $D(E_F)$ (in eV^{-1}) in both cases. Look up the values that you need.
d* Find the level spacing and density of states for a spherical gold cluster with radius R. See p129 of *introductuction to quantum mechanics* by Griffiths and use a computer to find the roots. Compare the results to the cubic case. Does the exact shape of the system matter when calculating the density of states?

Lecture 2: Length scales, transport regimes and classical conduction

1a Why is the Fermi wavelength much larger in a semiconductor compared to a metal?

b Semiconductors are often described in terms of the carrier concentration n and mobility μ. They are related to the conductivity by: $\sigma = ne\mu$. Express the elastic mean free path in terms of μ and n for 1,2 and 3 dimensions.

c What is the mean free path of a 2DEG with $n = 4 \cdot 10^{15}$ m^{-2} and $\mu = 10^5$ cm^2/Vs?

d Calculate the cyclotron radius when a perpendicular magnetic field $B=1$ T is applied to a GaAs 2DEG.

2 The phase coherence time τ_φ indicates how long an electron can remember its phase. How far does an electron get in this time? Consider the case with and without elastic scatterers in the material. Which relation separates the two cases?

3 A diffusive bar with width W = 0.13 μm is placed in a perpendicular magnetic field and the resistance is measured.

a The magnetic field is small. What happens with the resistance of the sample when the magnetic field is increased?

b For which magnetic field is the cyclotron radius equal to the width of the bar?

c What happens with an electron when the magnetic field is much higher that the field calculated in (b) ?

d Sketch the resistance versus magnetic field and compare this with the measurement by Thornton *et al* (Phys. Rev. Lett. 63, 2128, 1989)

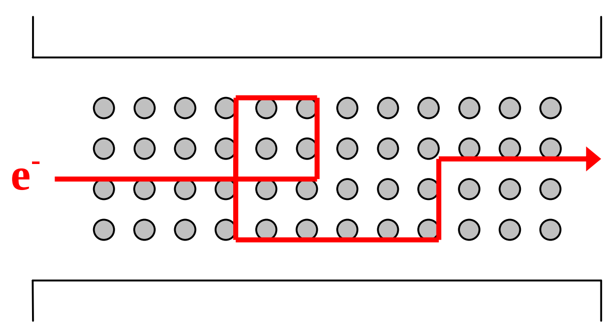
Lecture 3: Phase coherent transport 1

1a What is weak localization?

b Sketch the resistance of a device in which weak localization (WL) occurs versus the magnetic field. Which parameter can be extracted from such a measurement?

c We make channels in a 2DEG with an elastic mean free path of 1 μm and a phase coherence length of 100 nm. For which channel L lengths can we observe WL:
L = 1 cm, L = 10 μm, L = 300 nm L = 30 nm?

2 Suppose that we can make elastic scatterers with great accuracy, and we place them in a 2DEG in a square lattice. Electrons come in from one direction and when they hit a scatterer they either turn 90^o or just continue their way. The spacing between the scatterers is of the same order as l_φ. Sketch the magneto-resistance curve.



Lecture 4: Phase coherent transport 2

1a Express the flux quantum in fundamental constants. How large is it?

b What is the periodicity (in Tesla) of AB oscillations in a ring with a diameter of 1.0 μm and a multi-walled carbon nanotube with a diameter of 20 nm? Can AB oscillations be observed in a 2 nm single walled nanotube when using 12 T superconducting magnet?

c When you measure the magneto-resistance of an Aharonov-Bohm ring, you observe oscillations with both one and one-half of a flux quantum. What are the origins of both oscillations?

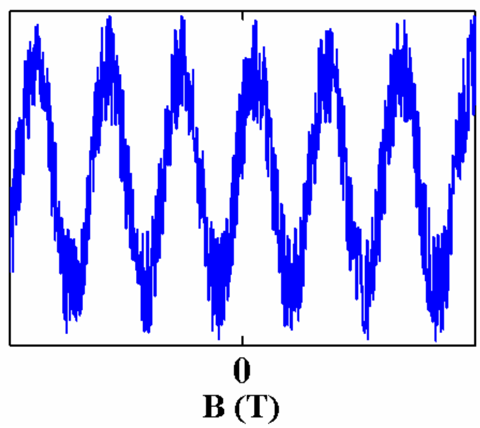
2a A metallic ring has a diameter of 0.5 μm, a thickness of 10 nm and a width of 20 nm. What are the periodicities of the AB effect when electrons travel along the outer and inner edge? Where is the AB effect (ΔR) larger, at small or large magnetic fields?

b* After performing a magneto-resistance measurement, the Fourier transformation of the data is used to find the periodicity. What happens with the *h/e* peak when only a finite range of magnetic field is measured?

c The two arms in the ring are almost, but not exactly the same. With which length scale do you have to compare the difference: L_e, L_φ or λ_F?

d You measure the magneto-resistance (see figure). The minimum in resistance does not lie at zero magnetic field. Can you explain this?

e With AAS oscillations, can you have a minimum, a maximum or any value in between at zero magnetic field?



3a What is universal about UCF?

b UCF is measured in a rectangular bar with length *L*, width *W* and thickness *H*. Express the square resistance $R_\square$ in the resistivity and dimensions of the device.

c Show that the resistance of the bar can be obtained by "counting squares".

d A samples with a resistance of 100 Ω shows UCF. How large are the fluctuation in the resistance, both in Ω and as a percentage.

Lecture 5: Ballistic transport

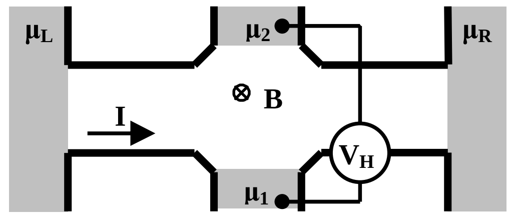
1a What is the order of the length-scales for classical ballistic transport: L, λ_F, l_e and l_φ?

b How does the resistance of a channel change when the length is increased for a quasi- and true ballistic channel?

c A ballistic nanotube is measured in a four-terminal geometry. What is larger, the two-terminal or the four-terminal resistance and what causes the difference.

d First the two-terminal resistance is measured. What values for the four point-measurements can you expect?

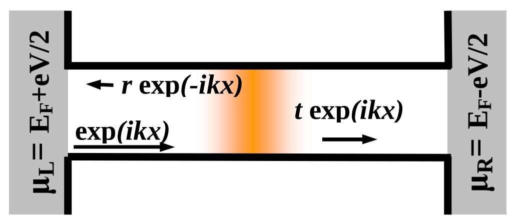
The experiment has indeed been done. You can find the answers to this exercise in B. Gao *et al*, Phys. Rev. Lett. **95**, 196802 (2005)



2a Express the chemical potential of the left reservoir (μ_L) of the Hall bar shown on the right in terms of the voltage drop between the left and right lead and μ_R.

b A perpendicular magnetic field is applied to the ballistic Hall bar. Sketch the trajectories of electrons when the cyclotron radius is much smaller than the width of the bar. What are the chemical potentials μ_1 and μ_2 and the Hall resistance V_H/I in this case?

c Sketch the electron trajectories for lower magnetic fields. What happens with the Hall resistance when the field is decreased? Is it possible that the Hall resistance becomes negative (when $B > 0$)?



3 A one dimensional conductor contains a single scatterer with transmission t and an electron with wavefunction $\psi(x) = \exp(ikx)$ is sent into the wire.

a The probability current J determines how fast the probability of finding the electron in the right reservoir changes and is given by†:

$$J = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right).$$

Express the probability current in terms of t and the velocity of the electron.

b What is the density of states of a one dimensional conductor? How many electrons flow in the channel from the left reservoir when a small voltage V is applied between the left and right reservoir?

c Calculate the current through the wire. Does it depend on the electron velocity?

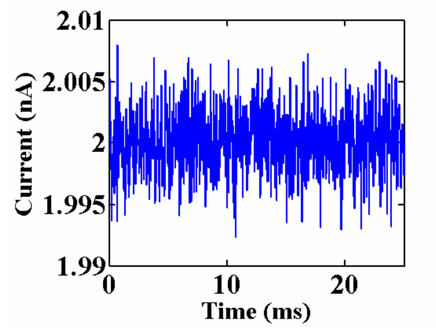
d When the width of the wire is larger than λ_F, more than one conduction channel is open. For each channel, the transmission can be different. Show that the conductance of the wire is given by:

$$G = \frac{2e^2}{\hbar}\sum_n |t_n|^2.$$

e* Assume that the scatterer is a rectangular potential of height $V_0 > E_F$ and size d. Calculate the (energy-dependent) transmission coefficient $t(E)$ and the make a plot of the voltage dependence of the differential conductance.

† See for example p13 of *introductuction to quantum mechanics* by Griffiths

4a The measured current through a point contact with two available channels with a voltage V = 20 μV is applied is shown. What is conductance in Siemens and in units of G_0?



b The noise is also measured and a value of S_I = 2.0 ·10^{-28} A^2/Hz is found. The noise is related to the transmission of the channels T_n by:

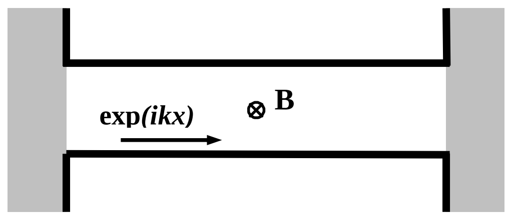
$$S_I = 2eI \cdot \frac{\sum_n T_n(1-T_n)}{\sum_n T_n}.$$

What are the transmissions T_1 and T_2 of the two channels?

c* What is the bandwidth of the current meter that was used? How large is the thermal (Johnson) noise in pA of this point contact at a temperature of 4 K?

d The shot noise in a resistor is $S_I = 2eI$. Is the noise in a quantum point-contact larger or smaller than this value?

5* Electrons in a high magnetic field form Landau levels. In this exercise it will be shown that this can be described quantum mechanically as a harmonic oscillator.



a What is the Hamiltonian for an electron in a uniform magnetic field? Expand the squared term. Pay attention to the order of the gradient and vector potential $\boldsymbol{A}$.

b The $\boldsymbol{B}$ field is applied in the z-direction, perpendicular to the 2DEG. What is the general form of the vector potential? Note that that there is a gauge freedom.

c The electron lives in a 2DEG, where the (scalar) potential only depends on the direction perpendicular to it $V(x,y,z) = V(z)$. Show that with the gauge choice

$$\mathbf{A} = -B_z/2\hat{\mathbf{x}} + B_z/2\hat{\mathbf{y}}$$

the Hamiltonian can be separated in $H = H_{xy}(x,y) + H_z(z)$, so Schrödinger equation can be solved by inserting $\Psi(x,y,z) = \psi(x,y)\chi(z)$. We assume that the first excited state of H_z is far above E_F.

d Show that $\psi(x,y) = u(y)e^{ikx}$ represents an electron moving in the x-direction. Show that the Schrödinger equation for $u(y)$ is:

$$\left(-\frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial y^2} + \frac{\hbar^2 k^2}{2m^*} - \frac{\hbar keBy}{m^*} + \frac{e^2B^2y^2}{2m^*}\right)u(y) = \left(E - E_z^0\right)u(y).$$

e Show with a change of coordinates $y \rightarrow \eta$ this can be written as the Schrödinger of an harmonic oscillator:

$$\left(-\frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial \eta^2} + \frac{1}{2}m^*\omega^2\eta^2\right)u(\eta) = \left(E - E_z^0\right)u(\eta).$$

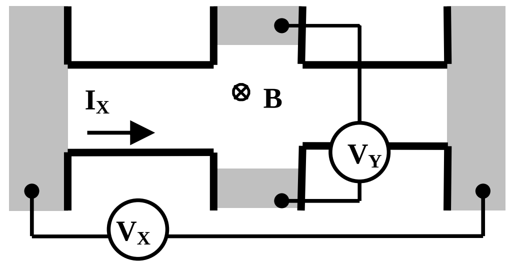
What is the frequency ω expressed in B? Calculate the level spacing in eV for m^* = 0.07·m_e, B = 1 T and k = 0.16 nm^{-1}. How does it compare to the Zeeman energy?

f In the absence of a magnetic field, the electron average y position of the electron is the middle of the 2DEG. Show that when Landau levels are formed, the electron is closer to one of the edges of the 2DEG.

6 In the lectures we have encountered the Hall effect three times as:

- The classical diffusive Hall effect,
- The classical ballistic Hall effect,
- The (integer) quantum Hall effect.

What are the differences and similarities in these different effects? Hint: what is measured, what are the relevant length scales and make sketches of electron paths.



7a The resistance R_{XX} of the two dimensional bar shown in the figure is measured. Express the resistance R_{XX} in the resistance of a square $R_\square$ and the dimensions of the bar. How does the resistance depend on the length L?

b In the classical diffusive limit, what is the force $\boldsymbol{F}$ on an electron in an electric and magnetic field? How are the average velocity of the electrons and the current density related?

c Now the Hall voltage V_Y is measured and divided by the applied current I_X to obtain the Hall resistance R_{XY}. Can a net current flow in the y direction? Use the answer to b to express the R_{XY} in the 2D electron concentration.

8a We now turn to the integer quantum Hall effect. What is the spacing in energy between two adjacent Landau levels and calculate the value in eV for a field of 1 T? Sketch the density of states for $B = 0$ and for high magnetic fields.

b The number of electrons in a 2DEG that is not connected to leads, does not change when the magnetic field is switched on. How many electrons are there in a single Landau level? Is the Landau level just below the Fermi energy completely filled? Sketch the magnetic field dependence of the Fermi energy.

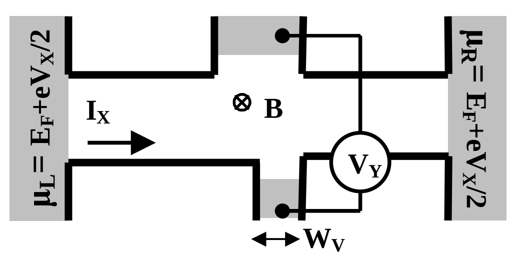
c When the 2DEG is connected to the leads, electrons are inserted due to the difference in chemical potential between the metal and the 2DEG. The chemical potential of the leads does not change, so how many electrons are in the highest occupied Landau level?

d Use the number of filled Landau levels to calculate the 2D electron concentration and insert this into the answer to 6c. Is the result what you expect? Was the derivation done correctly or have we been cheating?

9a The extent of the wavefunctions in a harmonic oscillator are related to the zero point uncertainty:

$$u_0 = \sqrt{\hbar/2m\omega}.$$

Its value for a Landau level is called the magnetic length. Express it in terms of fundamental constants and the strength of the magnetic field.



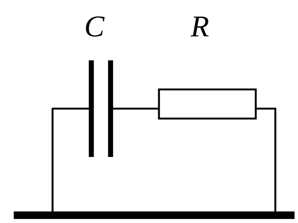
b The width of one the voltage probe leads is small: $W_V = 50$ nm. Calculate the magnetic field dependence of the Hall resistance for $E_F = 5$ eV and $m^* = m_e$. Does the answer depend on the value of V_x?

Single electron tunneling and Coulomb blockade

1a What is the relevant energy-scale for Coulomb blockade?

b How large is the capacitance (to infinity) of a sphere with radius r = 1 nm. Compare this to $k_B T$ at room temperature and at T = 50 mK.

2a An amount of charge Q_0 is placed on a capacitor with capacitance C. There is a resistance R to ground. What is the time-dependent charge $Q(t)$ on the capacitor?
What is the characteristic time-scale for this problem?



b The Heisenberg uncertainty principle states that the energy of an electron is ill defined when the electron stays in a state only for a short time:

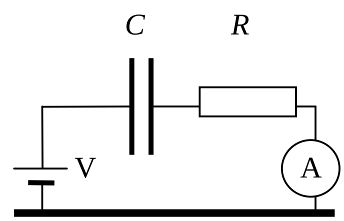
$$\delta E \cdot \delta t \geq \hbar/2.$$

What is the uncertainty in energy for the system discussed in a?

c This uncertainty has to be compared with the charging energy $E_C = e^2/2C$. In which case can Coulomb blockade be observed: $\delta E >> E_C$ or $\delta E << E_C$?

d Which relation should hold for the resistance to observe Coulomb blockade? Do you recognize this value? Does it depend on the capacitance?

e The network is connected to a voltage source and a current meter. Draw the IV characteristics for R = 1 kΩ and R = 100 kΩ.

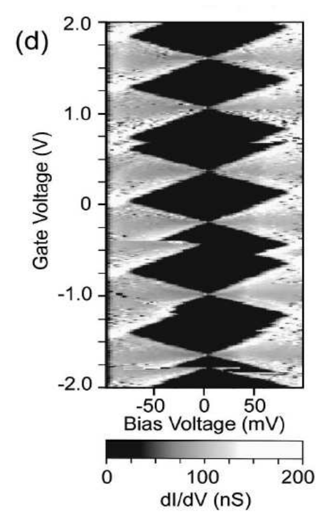


3a The stability diagram shown on the right is measured on a small gold grain. (K.I. Bolotin *et al.*, APL **84**, 2004, 3154). What is the charging energy and the total capacitance of the grain?

b Use the slopes of the diamonds to find the gate coupling C_g/C_{tot}. What are the gate, source and drain capacitances?

c At V_g = -0.5 V a switch occurs. How much is the change in the induced (offset) charge? Is this an integer multiple of e?

d Suppose that the switch is due to the charging of another island nearby by a single electron. What is the capacitance between the two islands?



4a* A metallic sphere with radius R is placed at the origin. An electron is placed at $r=a>R$. Use the method of image charges to calculate its potential energy. Sketch the charge distribution on the sphere. Calculate the difference in energy for the electron located at r = ∞ and when the electron is located on the sphere. Is the difference positive of negative?

b What is the electrostatic potential of the sphere? Calculate its capacitance.

c Repeat 3a for the situation where an electron is already on the sphere.

d Repeat 3a for the situation where the sphere is grounded.

5a What is the total energy of N electrons on a large metallic island? Take both the charging energy and gate potential into account.

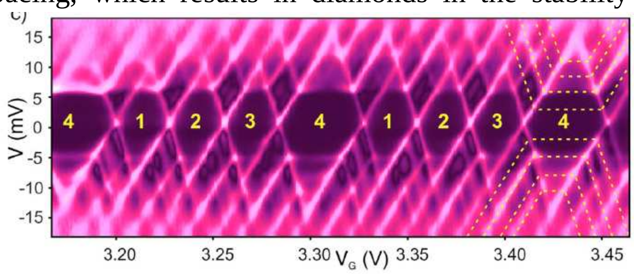
b Now the size of the island is made much smaller and the level-spacing becomes important. What is the total energy $U(N)$ in this case?

c What does the word “chemical potential” mean? How is it defined when only a small number of electrons is in the system?

d Calculate the chemical potential for the total energy in a and b. Is it the same for each electron?

e Calculate the gate voltages of the charge degeneracy points.

6a Electrons that tunnel to a quantum dot have to pay the charging energy and the charging energy and the level spacing, which results in diamonds in the stability diagram. The stability diagram shown below is measured here in Delft in a carbon nanotube quantum dot (Sapmaz *et al*. Phys. Rev. B **71**, 153402, 2005). Find the addition energies for each of the four different diamonds.



b There are many more lines visible in this stability diagram than one would expect for a simple quantum dot. Lines running parallel to the diamond edges can be used to find values for the energy difference between the ground state and excited states in a given charge state. Use a sketch of an energy diagram of the leads and the dot to explain how this works.

c The band structure in a metallic nanotube is linear and given by $E(k) = v_F hk/2\pi$. Calculate the levelspacing for a nanotube with length L = 350 nm. Which lines would correspond to this energy?

d When the effect of interactions between the electrons is on the dot neglected, each level has a four fold degeneracy (2x due to spin, 2x due to clockwise/anticlockwise). Find the charging energy of this nanotube from the sizes of the diamonds.

e What is the gate capacitance of the tube? Calculate the length of the tube using the equation for the gate capacitance in the slides. Is this in agreement with the length found from the levelspacing?

Nanomechanics

1a What is the difference between stress and strain? Explain what the Young's modulus and Poisson's ratio are. What is tension and what is bending rigidity?

b A bar with original dimension *LxWxH* is subject to a force F as shown on the right. The Young's modulus is E and the Poisson ratio ν. What are the dimensions of the beam after the force has been applied?

c When sound travels through the bar, slices will alternating be compressed or extended, both in space and time, resulting in travelling waves. The density of the beam is ρ. Find the (longitudinal) speed of sound of the bar. Does it depend on the dimensions?

d Different material have different mechanical properties. Look up the values to fill in this table with properties of materials often used for NEMS.

Material	E (GPa)	ν (-)	ρ (kg/m^3)	v_L (m/s)
Silicon				
Silicon Nitride				
Silicon Carbide				
Diamond				
Graphite (in-plane)				
IndiumPhosphide				

e Look up the crystal structure of graphite. Are its mechanical properties the same in all directions? When comparing the mechanical properties of carbon nanotubes with those of graphite, would you take the in-plane or out-of plane properties of graphite?

2a Bending modes of a beam are described by the Euler-Bernoulli equation:

$$\rho A \frac{\partial^2 u}{\partial t^2} + D \frac{\partial^4 u}{\partial x^4} = 0.$$

How does the bending rigidity D depend on the width W, height H and length L of a rectangular beam?

b Euler-Bernoulli equation looks similar to that of a simple harmonic oscillator, but now there is an x-dependence of the displacement u. Sketch the shape of the first three eigenmodes of doubly and singly clamped beams. Which one has the first mode with the highest frequency?

c Plug a function $u(x,t)=T(t)X(x)$ in this equation and move all terms that depend on t to the left. Put all terms that depend on x right of the equal sign. Show that both sides is a constant λ, i.e. they are independent of x and t. Write down the two resulting equations.

d Solve the equation for $T(t)$ and show with $\lambda = -\omega^2$ the equation of a simple harmonic oscillator is obtained.

e Show that $X(x) = a\cos(kx) + b\sin(kx) + c\cosh(kx) + d\sinh(kx)$ with $-k^4 = \lambda$ is the (general*) solution to the equation for the spatial dependence.

f* Solve the equation for $X(x)$ for a singly ($X(0)=X'(0)= X''(L)=X'''(L)=0$) and a doubly ($X(0)=X'(0)= X(L)=X'(L)=0$) clamped beam to obtain an expression that k should satisfy. Solve the first 3 roots of this equation numerically and plot $X(x)$. Compare the outcome with the answer to b.

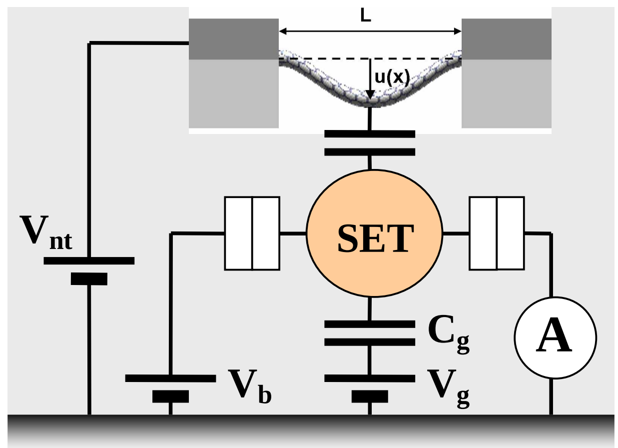
3a One of the important problems with NEMS is that not only the devices are small, but that the displacements are even smaller. Calculate the frequency of the first bending mode for a 200 nm long suspended carbon nanotube (take r = 0.7 nm, E = 1 TPa, ρ = $1.3\cdot10^3$ kg/m^3 and $I = \pi r^4$). Below which temperature is the resonator in it's groundstate?

b The mass appearing in the relation for the eigenfrequency is not equal to the total mass m of the nanotube:

$$f = \frac{1}{2\pi}\left(\frac{k}{m_{\text{eff}}}\right)^{1/2},$$

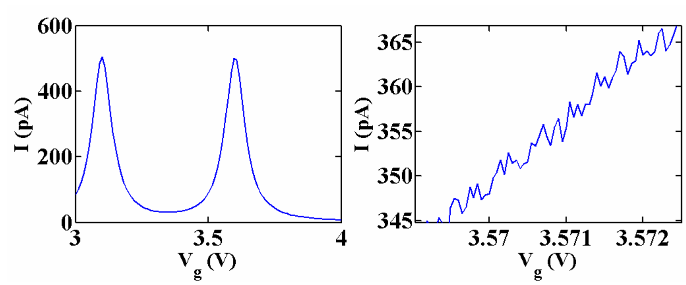
where $m_{\text{eff}} = 0.735\cdot m$. Why is the effective mass lower than the total mass? Find the springconstant of the nanotube. What is the thermal noise amplitude of the nanotube at room temperature and at 18 mK? How large is the zero-point motion of the resonator?

c These very small displacements can only be observed with special detection mechanisms. A single electron tunnelling device (SET) is placed 200 nm from the nanotubes and a voltage V_{nt} is applied to the nanotube. Use the expression for the capacitance between a nanotube and a plate to estimate the capacitance that couples the nanotube and the SET. Expand the answer for small displacements u.

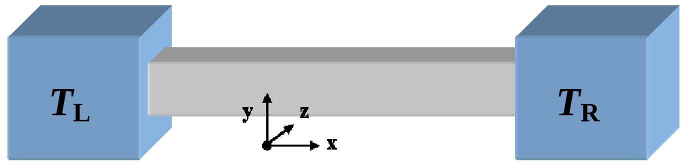


d Sketch the gate dependence $I(V_g)$ of the SET for a small bias V_b. What happens with the current when the nanotube is moving? Indicate in the sketch at which gate voltage the SET is most sensitive to changes in the tube's position. What happens with the sensitivity when V_{nt} is increased?

e First no voltage is applied to the nanotube and the SET is characterized. Some of the measurements are shown on the right. Furthermore, it is found that the source and drain capacitances can be neglected. Now the coupling between the resonator and the SET is switched on by applying V_{nt} = 4 V. Can the zero-point motion be detected with this detection scheme?



4a A rectangular bar with dimension LxWxH is connected to two large anchoring points which have different temperatures T_L and T_R. This will result in a net heat current flowing through the bar. The bar is made from silicon nitride, which is an isotropic and insulating material. Which contribution will be dominant, electron or phonon heat conduction[‡]?



b Phonons displace the volume elements of the beam from their initial position. How many phonon types does the beam have? The displacement field $\boldsymbol{u}(x,y,z,t)$ with components u_i is governed by a wave equation:

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \sum_{jk\ell} E_{ijk\ell}\frac{\partial^2 u_\ell}{\partial x_j \partial x_k}.$$

Here, ρ is the density and $\boldsymbol{E}$ is the elasticity tensor. Look up which of the elements of the elasticity tensor are zero for an isotropic material and express the other elements in terms of the Young's modulus and Poison ratio. Show that

$$u_i = e^{-i(\omega t - kx)} \cdot \chi(x,y)$$

is a solution to this equation. Which way is the does the wave travel for positive frequency and wavevector? Given the number of

c What are the boundary conditions at the sides (not the anchoring points) for the displacement? Show that this leads to the formation of subbands, just as in the case for ballistic electrons. Give an estimation for the energy at which the second (n=2) subband begins, without actually solving χ completely.

d* Show that the energy flow carried by a phonon is given by:

$$I_E(x,t) = -\sum_{jk\ell} E_{xjk\ell}\int_{W,H}\frac{\partial u_j^*}{\partial t}\frac{\partial u_\ell}{\partial x_k}\,\mathrm{d}y\mathrm{d}z.$$

e Insert the solutions for $\boldsymbol{u}$. What is the occupation of a phonon-mode with angular frequency ω travelling to the right? Is this the same for left-moving phonons. Compare this situation with that of electrons.

f Now show that the net heat current is given by the Landau-Buttiker expression:

$$I_{\mathrm{E,net}} = \frac{1}{h}\sum_n\int_{E_n,0}^{\infty} D_n(E)[n_L(E) - n_R(E)]\mathrm{d}E.$$

Show that the thermal conductance K due to the n=1 subband of one type of phonons for a small temperature difference is given by[§]:

$$\kappa = \frac{\kappa_Q}{T} = \frac{I_{\mathrm{E,net}}}{T^2},\ \text{with}\ \kappa_Q = \frac{\pi k_\mathrm{B}^2}{6\hbar},$$

and calculate the value of the universal thermal conductance quantum κ_Q.

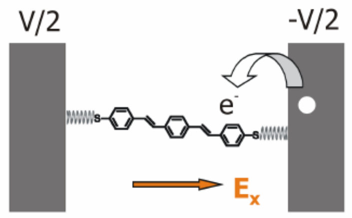
g When the temperature is increased, more than one subband of each type can contribute. For electrons higher subbands lead to a stepwise increase in the conductance. Is this the also the case for phonons?

‡ This exercise is based on the discussion in M. Blencowe, *Phys. Rep.*, **395**, 2004

§ Use the identity:

$$\int_0^\infty \frac{x^2 e^x}{(e^x-1)^2}\mathrm{d}x = \frac{\pi^2}{3}$$

5a A molecule is connected to two electrodes and a bias voltage is applied, which results in an electric field. The bonds between the molecule and the contacts are modelled as springs. The effective spring constant is k = 70 N/m and the mass of the molecule is $M = 1.2 \cdot 10^{-24}$ kg. At which frequency does the molecule vibrate and what is the levelspacing for this harmonic oscillator?



b An electron tunnels onto the molecule. Find the displacement of the molecule Δx for E_x = 1 V/nm. Compare this to the zero-point motion x_0 of the molecule and the thermal motion x_{rms} at 300 K and 18 mK.

c The electron-phonon coupling is defined as $\lambda = \Delta x/2x_0$. Find its value.

d Find the amount of energy gained by the displacement of the oscillator after tunneling and express this in terms of hf and λ. Explain that this leads to a shift of the charge degeneracy points in the stability diagram compared to the case where the molecule is fixed.

e* The quantum states of the entire molecule are the product of the electronic and the nuclear states. Close to the first charge-degeneracy point, there can only be zero or one electron on the molecule, so the electronics states are labeled with n=0 and n=1. Quantum states of the molecule are:

$$|\chi(x), n = 0\rangle\,,$$
$$|\chi(x), n = 1\rangle\,.$$

Write down the Hamiltonian for the harmonic oscillator, i.e. the nuclear part of the total Hamiltonian, for n=0 and n=1 separately and show that their eigenstates are different and that they are given by:

$$|\chi_\ell(x), n = 0\rangle \text{ and } |\chi_\ell(x - \Delta x), n = 1\rangle\,.$$

f* Because the atoms are much heavier than an electron, the nuclear wavefunction $\chi(x)$ stays the same just after the electron has tunneled. If $\chi(x)$ was the $|\ell\rangle$ eigenstate of the n=0 Hamiltonian, it will be a linear combination of eigenstates $|\ell'\rangle$ of the n=1 Hamiltonian. This means that energy has to be paid when $\ell' > \ell$, which has to be provided by difference between the energy of the electron in lead and the position of the electronic level ΔE. Sketch the lines in the stability diagram where a new transition becomes available.

g* Using Fermi's Golden rule, the rates of tunneling to and from the left and right lead can be calculated:

$$\begin{aligned} R^{L,R}_{\ell,0\to\ell',1} &= \Gamma_{L,R}/\hbar|\langle\ell|\,\ell'\rangle|^2 f\big(\Delta E + \hbar\omega(\ell-\ell')\pm eV/2\big) \\ R^{L,R}_{\ell',1\to\ell,0} &= \Gamma_{L,R}/\hbar|\langle\ell|\,\ell'\rangle|^2\left[1 - f\big(\Delta E + \hbar\omega(\ell-\ell')\pm eV/2\big)\right]. \end{aligned}$$

This expression contains the overlap between the eigenstates of the oscillator in the different charge states. Use a Taylor series to prove that**:

$$\chi_\ell(x-\Delta x) = e^{-\Delta x\frac{\partial}{\partial x}}\chi_\ell(x) = e^{-\lambda(\hat a^\dagger - \hat a)}\chi_\ell(x).$$

h* Calculate the overlap's of the ground state $|\ell=0\rangle$ of the oscillator with $|\ell'\rangle$ and show that it is equivalent to the Poisson distribution. More difficult is the calculation of the entire overlap matrix $\langle\ell|\,\ell\rangle$, but it is possible to do. You can find the answer in McCarthy *et al*, PRB, **67**, 245415 (*2003*).

** Recall that the exponential of an operator is defined as a sum:

$$e^{\hat O} = \sum_{n=0}^{\infty}\frac{\hat O^n}{n!}.$$

Mesoscopic superconductivity

1a Name three properties of a superconductor. Are all of these properties relevant for mesoscopic physics?

b How small can a superconductor be?

2 On the interface between a normal metal and superconducting material (NS), electrons either have enough energy to overcome the gap of the superconductor or they have to be "converted" into Cooper pairs, before they can enter the superconductor.

a Why is an electron with E-$E_F > \Delta$ in the superconductor no longer called "electron", but instead "quasiparticle"?

b How many electrons are there in a Cooper pair? What is the total spin of a Cooper pair? Which type of statistics do Cooper pairs obey: Boltzmann, Fermi-Dirac or Bose-Einstein?

c Sketch the process of Andreev reflection (AR) as a cartoon.

d An NSN junction is made and a current is sent through it. An electron comes in from the left and AR occurs at the left interface. What will happen with the Cooper pair at the other side?

e When a SNS junction is made, the hole that is generated by AR at the right interface travels to the left interface. Explain the multiple Andreev reflection process. State the criterion for bound states.

f Does Andreev refection occur in a SIS (superconductor-insulator-superconductor) junction? Plot the IV curve and the differential conductance for such a junction.

3 Small islands of different materials are made and connected to source and drain electrodes and a gate. Unfortunately the boxes with the different samples fall on the ground. One is lost and the others are mixed and it is not possible to distinguish between the different samples anymore. The stability diagrams of the remaining SETs are measured anyway and the results are shown below. The two of the samples consisted of a 100 nm grain made of gold and aluminum. The other two were smaller: 20 nm in diameter. All measurements were performed at 50 mK. Can you say which sample was which? What experiment can be done to determine whether the sample with the small island is made of aluminum or gold?