AE4520: Advanced Structural Analysis

Strain in 2D and 3D

Mostafa Abdalla

Learning Objectives

- Understand the strain tensor
- Understand the small-displacement approximation
- Be able to derive the compatibility conditions of strains
- Relate the strain tensor to customary notions
  - Principal strains
  - Mohr’s circle of strain

Summary

- Deformation gradient
  - Maps vectors in un-deformed to corresponding vectors in deformed configurations
- Length (squared!) of vectors in the deformed configuration depends on a tensor (Cauchy tensor)
- The square-root of Cauchy tensor describes the “stretching” due to deformation
- Polar decomposition
  - Deformation is composed of a pure stretch and rigid rotation

Deformation Gradient

$$\begin{pmatrix} dx' \\ dy' \\ dz' \end{pmatrix} = \mathbf{F} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}, \quad d\mathbf{r}' = \mathbf{F}\, d\mathbf{r}$$

- Displacement gradient

$$\mathbf{H} = \begin{bmatrix} u_{,x} & u_{,y} & u_{,z} \\ v_{,x} & v_{,y} & v_{,z} \\ w_{,x} & w_{,y} & w_{,z} \end{bmatrix}$$

- Deformation gradient $\mathbf{F} = \mathbf{I} + \mathbf{H}$

Strain

- Length after deformation

$$ds'^2 = d\mathbf{r}^t \mathbf{C} d\mathbf{r}, \quad \mathbf{C} = \mathbf{F}^t \mathbf{F}$$

- Principal Strain directions

$$\left(\mathbf{C} - \lambda \mathbf{I}\right)\mathbf{e} = \mathbf{0}$$

- Polar decomposition

$$\mathbf{I} + \mathbf{H} = \mathbf{F} = \mathbf{R}\mathbf{U}, \;\; \mathbf{U} = \mathbf{U}^t, \quad \mathbf{C} = \mathbf{U}^2, \;\; \mathbf{R}^t \mathbf{R} = \mathbf{I}$$

- Green strain

$$2\mathbf{E}_g = \mathbf{C} - \mathbf{I}, \;\; \mathbf{E}_g = \boldsymbol{\varepsilon} + \tfrac{1}{2}\mathbf{H}^t \mathbf{H}$$

Small and Moderate Strains

- For small displacements

$$\mathbf{H} = \boldsymbol{\varepsilon} + \boldsymbol{\varphi}, \;\; \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^t, \quad \boldsymbol{\varphi} = -\boldsymbol{\varphi}^t$$

- For small strains moderate rotations

$$\mathbf{E}_g = \boldsymbol{\varepsilon} - \tfrac{1}{2}\boldsymbol{\varphi}^2$$

- Strains

$$\varepsilon_x = u_{,x} + \tfrac{1}{2}\left(\varphi_y^2 + \varphi_z^2\right) \quad \gamma_{yz} = v_{,z} + w_{,y} - \varphi_y\varphi_z$$

$$\varepsilon_y = v_{,y} + \tfrac{1}{2}\left(\varphi_z^2 + \varphi_x^2\right) \quad \gamma_{zx} = w_{,x} + u_{,z} - \varphi_z\varphi_x$$

$$\varepsilon_z = w_{,z} + \tfrac{1}{2}\left(\varphi_x^2 + \varphi_y^2\right) \quad \gamma_{xy} = u_{,x} + v_{,y} - \varphi_x\varphi_y$$

- Rotations

$$2\varphi_x = w_{,y} - v_{,z}, \;\; 2\varphi_y = u_{,z} - w_{,x}, \;\; 2\varphi_z = v_{,x} - u_{,y}$$

Strain compatibility

- Compatibility condition (2D)

$$\varepsilon_{x,yy} + \varepsilon_{y,xx} = \gamma_{xy,xy}$$

- In 3D we can have up to 6 different equations

- Only 3 are linearly independent