P-L-3

Mechanics

The Modelling Team
Department of Design Engineering
Faculty of Industrial Design Engineering
Delft University of Technology

Contents

- Systems
- Key components of Mechanical Systems
  - Spring
  - Damper
- Mass-Spring-Damper Systems
- Applications in product design

Aim

1. To develop a basic understanding of the properties of mass-spring-damper system and its applications in product designs — **Knowledge**

2. To demonstrate that such a system might be modeled so as to provide useful data for designs — **Insight**

3. To communicate with experts in their professional languages — **Communication**

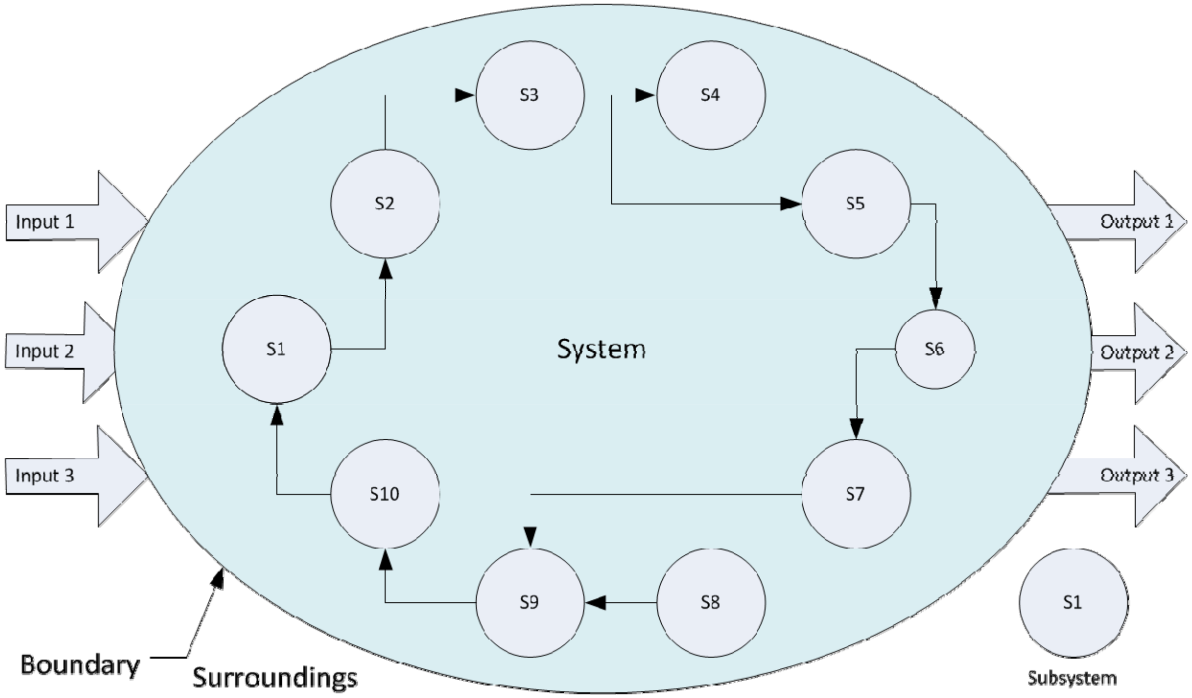
Systems

Repetition: What is a system?

System

System consists of a set of interacting or interdependent system components (or sub-systems)

- -Structure & interconnectivity
- -Boundary
- -Input & Output
- -Surroundings



Example of a system

Tyre

Tyre

Courtesy of http://www.cycle9.com/carrboro-chapel-hill-store/batavus-bicycles-lets-go-dutch/
Courtesy of http://www.batavus.nl

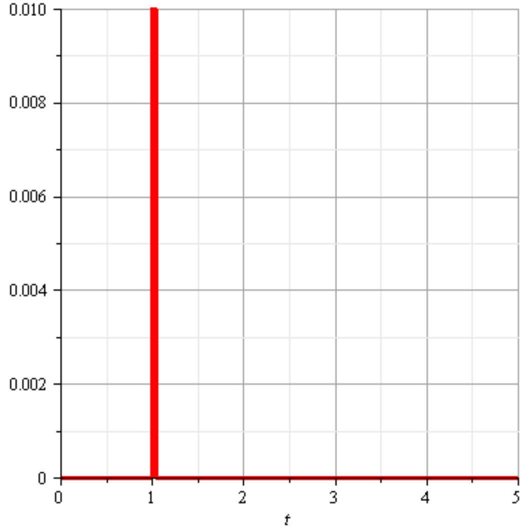
System response

Dirac function

$$\delta(t) = \begin{cases} \infty & t = t_0 \\ 0 & others \end{cases}$$

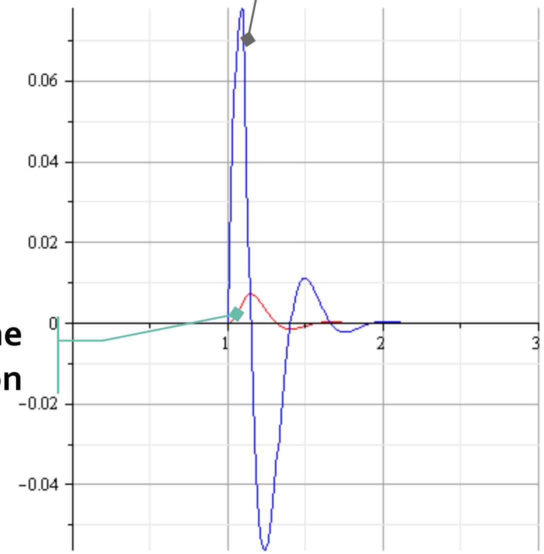
Maple®

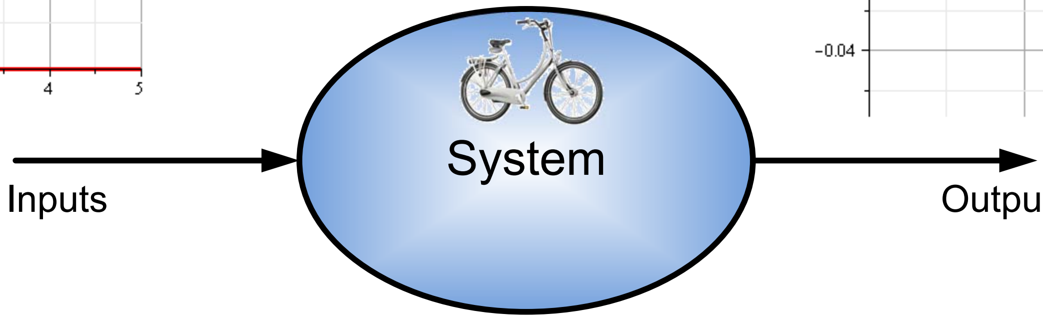
$$f := t \rightarrow \mathrm{Dirac}(t-1)$$



Velocity in the vertical direction

Displacement in the vertical direction





Boundary

Courtesy of http://www.batavus.nl
Courtesy of http://reviews.mtbr.com/blog/tag/Cycle-Solutions/

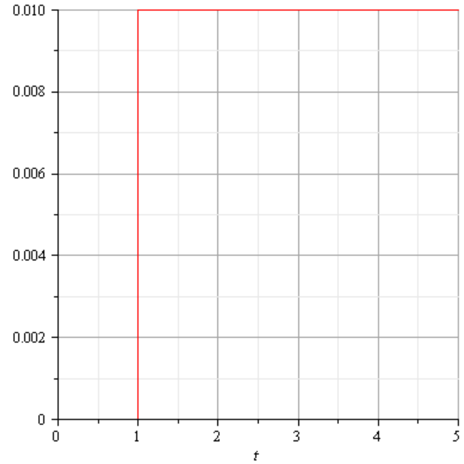
System response

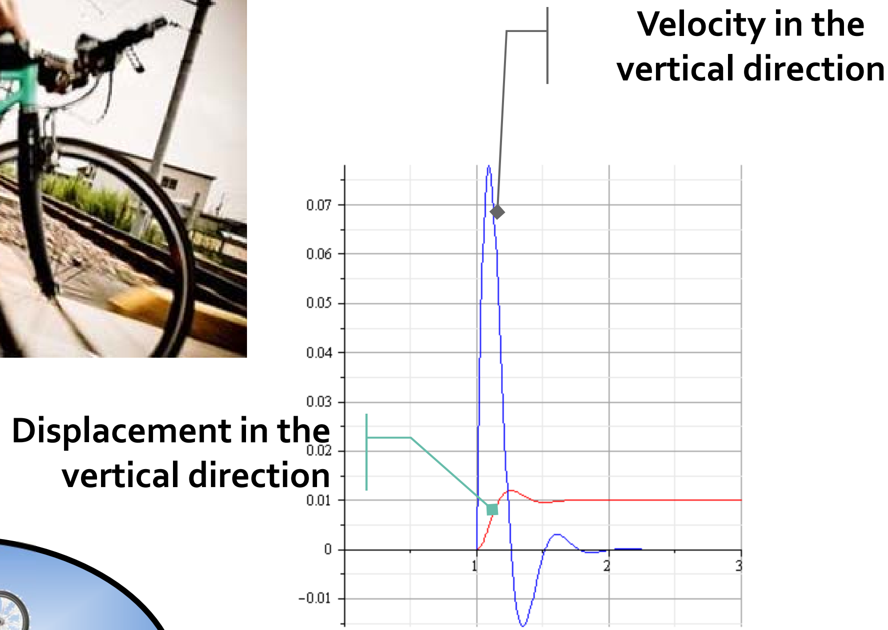
Step function

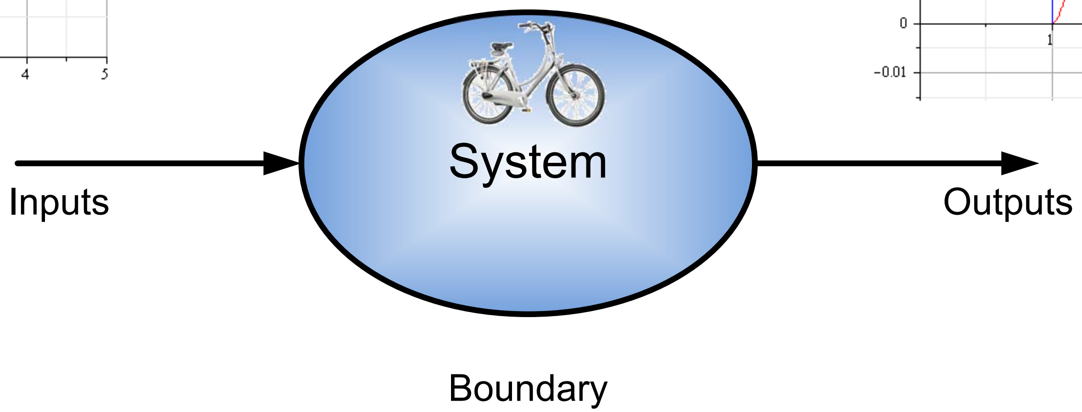
$$f(t) = \begin{cases} 0 & t < t_0 \\ undefined & t = t_0 \\ x_o & t > t_0 \end{cases}$$

Maple®

$$f := t \to \mathrm{Heaviside}(t - 1)$$

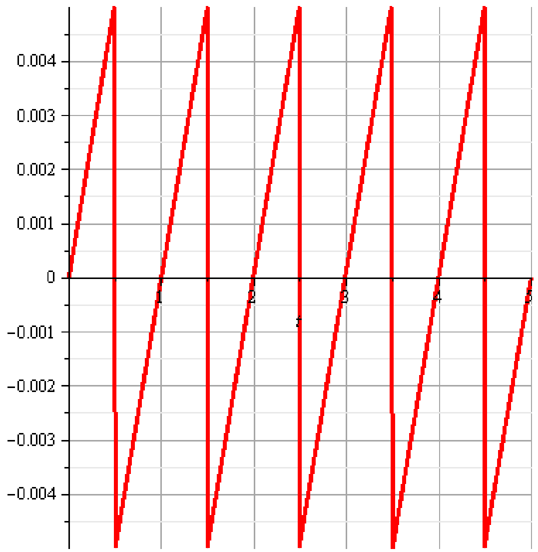


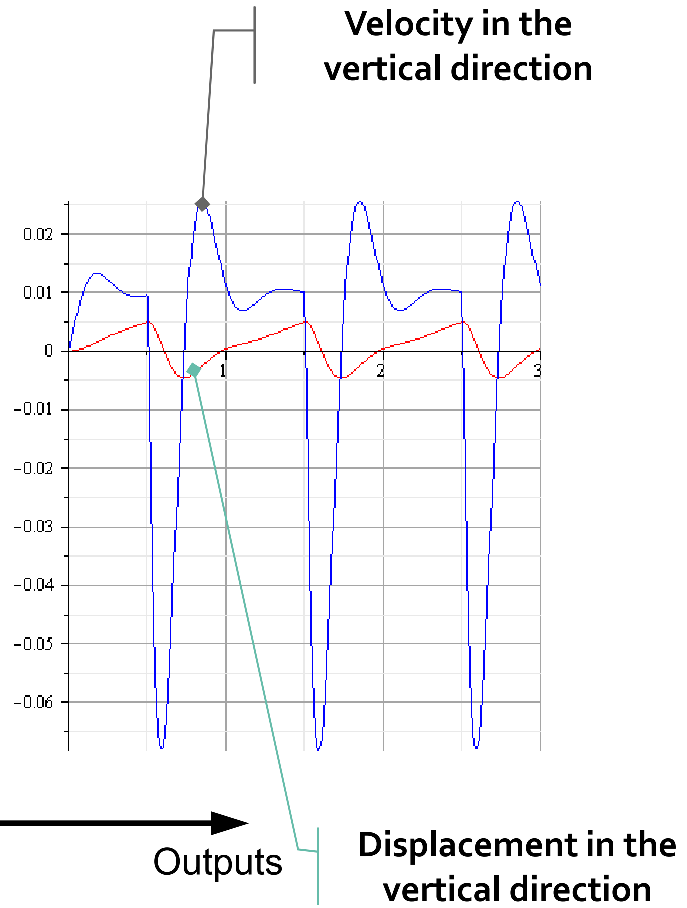


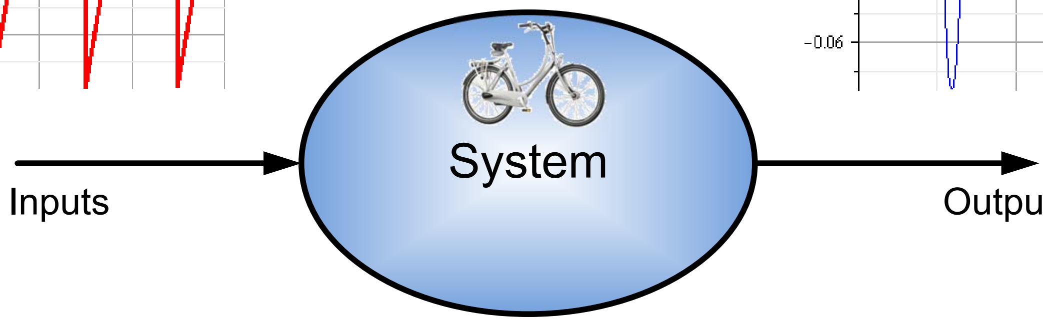


Courtesy of http://www.batavus.nl
Courtesy of http://www.utilitycycling.org/2009/08/cycling-services/

System response

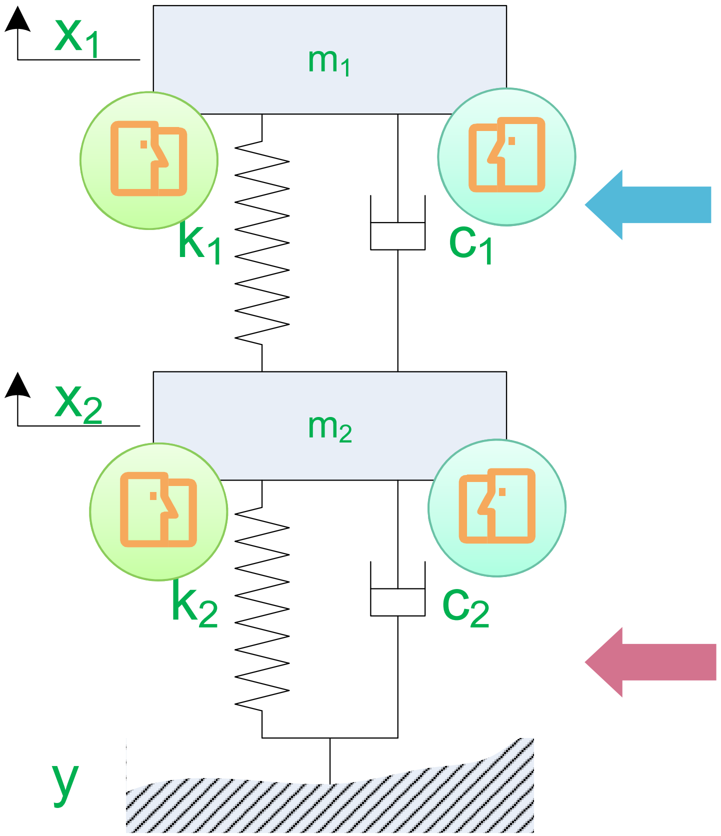






Courtesy of http://www.batavus.nl
Courtesy of http://www.utilitycycling.org/2009/08/cycling-services/

What are components of the bike's model?



Courtesy of http://www.batavus.nl

Components - Spring

Component - Spring

Spring

A spring is
an elastic object used to
store mechanical energy

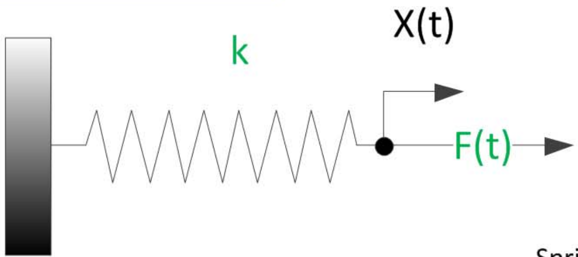
Ref. http://en.wikipedia.org/wiki/Spring_(device)

Model a spring

Spring – Linear

Courtesy of http://en.wikipedia.org/wiki/Spring_(device)

Sketch



Model

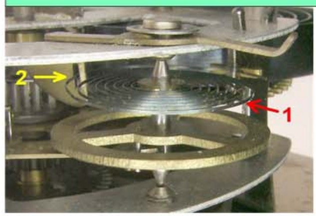
Force[N]

$$F(t) = -kx(t)$$

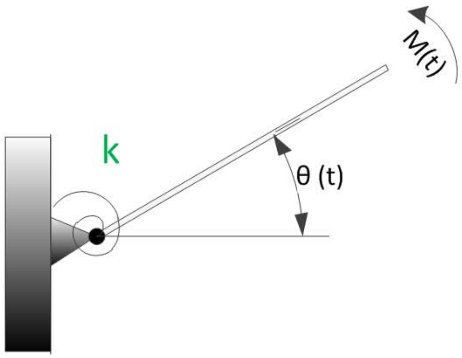
Spring Constant [N/m]

Displacement [m]

Spring – Angular



Courtesy of http://en.wikipedia.org/wiki/Spring_(device)



Torque[Nm]

$$M(t) = -k\theta(t)$$

Spring Constant [Nm/radian]

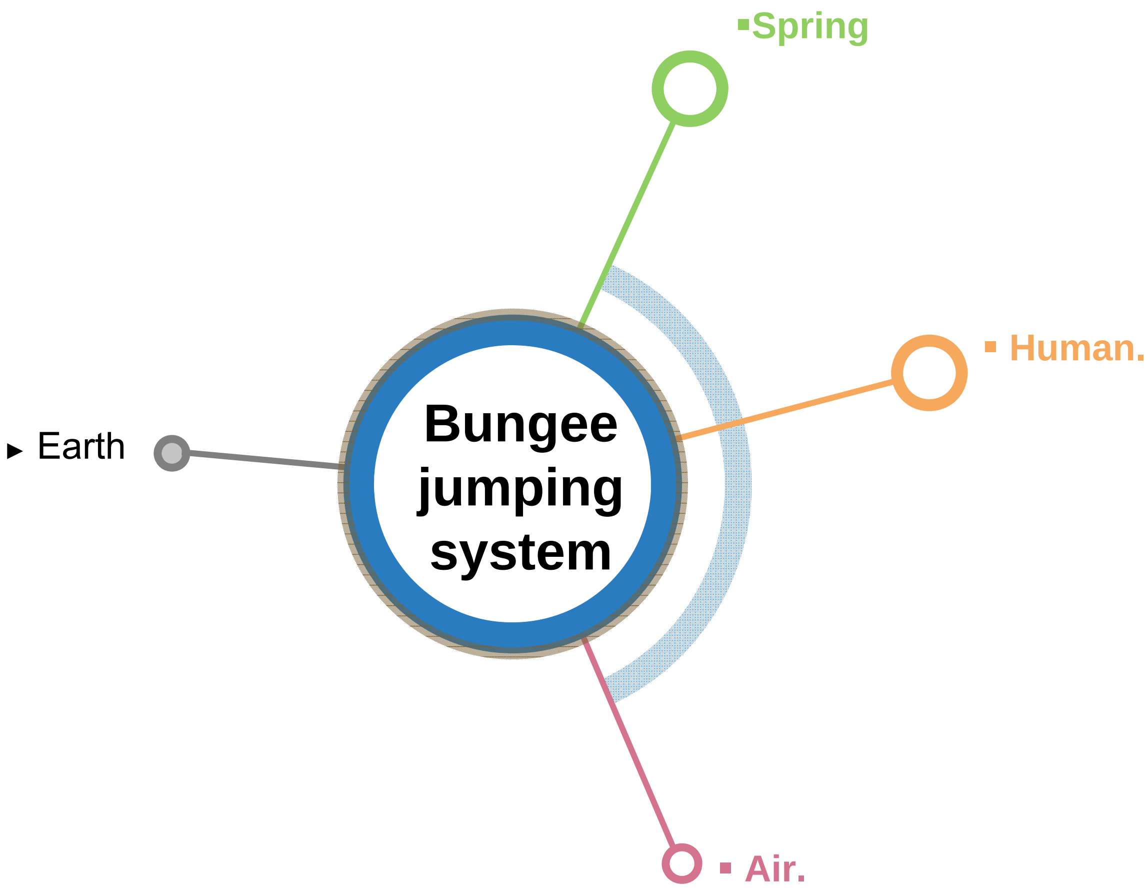
Angular displacement [radian]

Courtesy of http://en.wikipedia.org/wiki/Spring_(device)

Rehearsal: Bungee jumping

Courtesy of http://www.youtube.com/watch?v=sriUfHuHSXY

Components



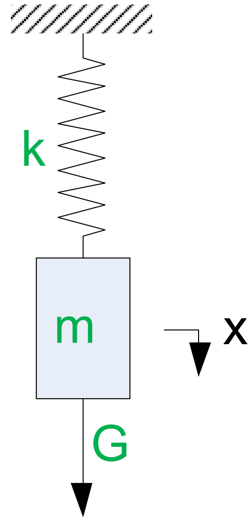
Cause-effect

Cause

Mass of the person	Gravity	Spring	Air friction

Effect

Inertia force	Stretch the spring	Pull the person back	Slow down the motion



$$-m\frac{dx^2(t)}{dt^2} + mg - k\begin{cases}0 & x(t) \le L \\ x(t) - L & x(t) > L\end{cases} - signum(\frac{dx(t)}{dt})\frac{1}{2}\rho_{air}Ac_d(\frac{dx(t)}{dt})^2 = 0$$

$$signum(x) = \begin{cases}-1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0\end{cases}$$

Modelling using Maple®

Suppose the displacement is x(t)

> restart :

> $equ := -m \cdot diff(x(t), t\$2) - k \cdot (\text{piecewise}(x(t) \le L, 0, x(t) > L, x(t)-L)) + m \cdot g - \text{signum}(diff(x(t), t)) \cdot \frac{1}{2} \cdot \rho_{air} \cdot c_d \cdot A \cdot diff(x(t), t)^2 = 0$

$$equ := -m\left(\frac{d^2}{dt^2}x(t)\right) - k\left(\begin{cases} 0 & x(t) \le L \\ x(t) - L & L < x(t) \end{cases}\right) + m\,g - \frac{1}{2}\,\text{signum}\left(\frac{d}{dt}x(t)\right)\rho_{air}\,c_d\,A\left(\frac{d}{dt}x(t)\right)^2 = 0 \quad \textbf{(1.1)}$$

>

Initial conditions

> $ics := x(0) = 0, D(x)(0) = 0;$

$$ics := x(0) = 0, D(x)(0) = 0 \quad \textbf{(1.2)}$$

The solution

Solutions

$> sol := dsolve(\{equ, ics\}, x(t), numeric, output = listprocedure);$

$$sol := \left[t = \textbf{proc}(t) \;\ldots\; \textbf{end proc}, x(t) = \textbf{proc}(t) \;\ldots\; \textbf{end proc}, \frac{d}{dt} x(t) = \textbf{proc}(t) \;\ldots\; \textbf{end proc} \right]$$

displacement

$> x := rhs(sol[2]);$

$$x := \textbf{proc}(t) \;\ldots\; \textbf{end proc}$$

speed

$> v := rhs(sol[3]);$

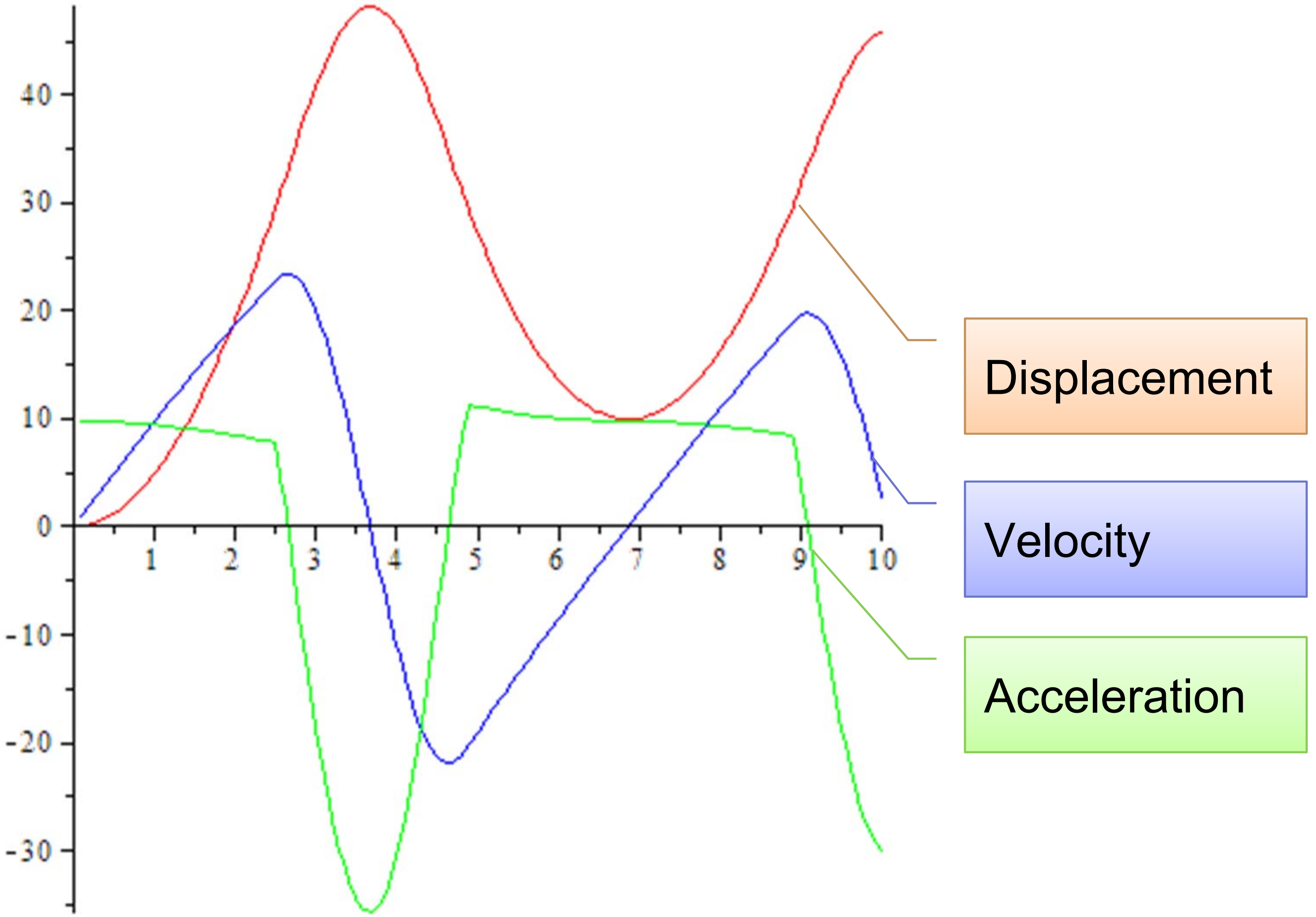
$$v := \textbf{proc}(t) \;\ldots\; \textbf{end proc}$$

acceleration

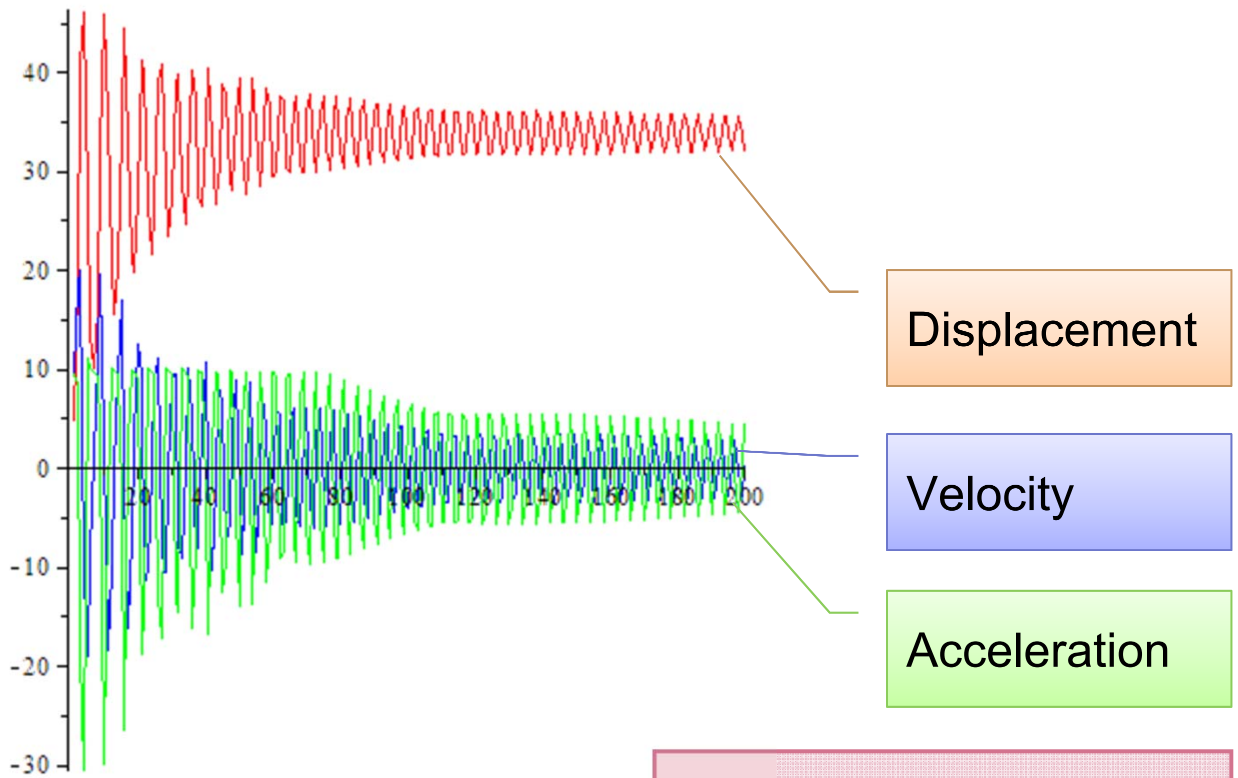
$> a := k \rightarrow fdiff(v(t), t = k);$

$$a := k \rightarrow fdiff(v(t), t = k)$$

The solution



The solution

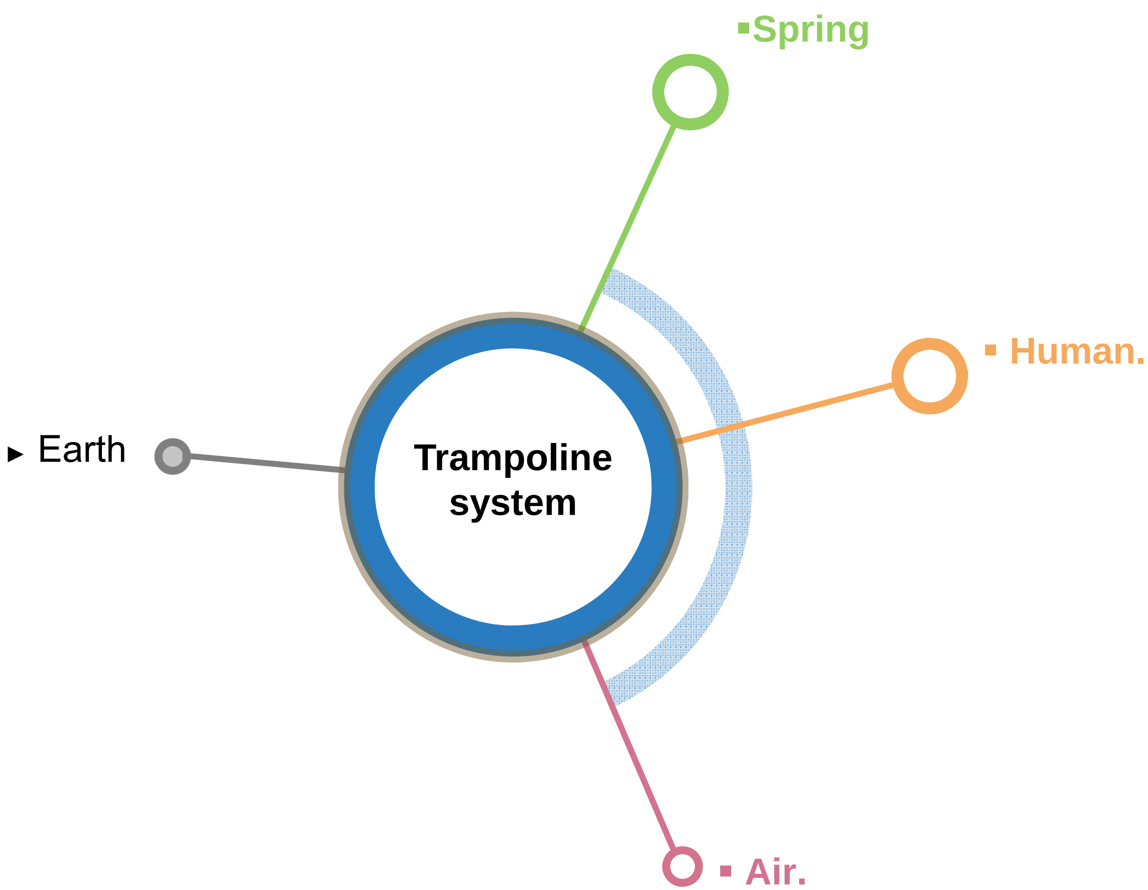


The amplitude is getting smaller, Why?

Is it different? - Trampoline

Courtesy of http://www.youtube.com/watch?v=pSazagYBCM8

Components



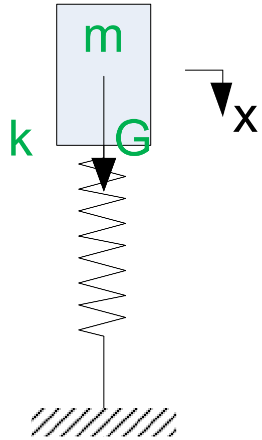
Cause-effect

Cause

Mass of the person	Gravity	Spring	Air friction

Effect

Inertia force	Stretch the spring	Push the person back	Slow down the motion



$$-m\frac{dx^2(t)}{dt^2} + mg - k\begin{cases} 0 & x(t) < 0 \\ x(t) & x(t) \geq 0 \end{cases} - signum\left(\frac{dx(t)}{dt}\right)\frac{1}{2}\rho_{air}Ac_d\left(\frac{dx(t)}{dt}\right)^2 = 0$$

$$signum(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

Modelling using Maple®

Trampoline

$\sum F = 0$

Inertia force + Force of Spring +Gravity+dragforce= 0

Suppose the displacement is x(t)

> *restart* :

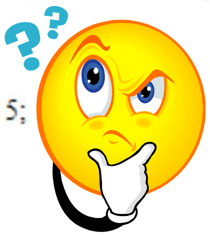
> $equ := -m \cdot diff(x(t), t\$2) - \mathrm{k} \cdot (\mathrm{piecewise}(\mathrm{x(t)} < 0, 0, \mathrm{x(t)} \geq 0, \mathrm{x(t)})) + \mathrm{m} \cdot \mathrm{g} - \mathrm{signum}(diff(x(t), t)) \cdot \frac{1}{2} \cdot \rho_{air} \cdot c_d \cdot A \cdot diff(x(t), t)^2 = 0$

$$equ := -m \left(\frac{d^2}{dt^2} x(t) \right) - k \left(\begin{cases} 0 & x(t) < 0 \\ x(t) & 0 \leq x(t) \end{cases} \right) + m\,g - \frac{1}{2}\,\mathrm{signum}\left(\frac{d}{dt} x(t) \right) \rho_{air}\, c_d\, A \left(\frac{d}{dt} x(t) \right)^2 = 0$$

>

Initial conditions

> $ics := x(0) = 0, \mathrm{D}(x)(0) = 5;$



$$ics := x(0) = 0, \mathrm{D}(x)(0) = 5$$

The solution

Solutions

$> \ sol := dsolve(\{equ, ics\}, \mathrm{x}(t), numeric, output = listprocedure, maxfun = 10000000);$

$$sol := \left[t = \mathbf{proc}(t) \ \ldots \ \mathbf{end\ proc}, x(t) = \mathbf{proc}(t) \ \ldots \ \mathbf{end\ proc}, \frac{\mathrm{d}}{\mathrm{d}t} x(t) = \mathbf{proc}(t) \ \ldots \ \mathbf{end\ proc}\right]$$

displacement

$> \ \mathrm{x} := rhs(sol[2]);$

$$x := \mathbf{proc}(t) \ \ldots \ \mathbf{end\ proc}$$

$>$

speed

$> \ v := rhs(sol[3]);$

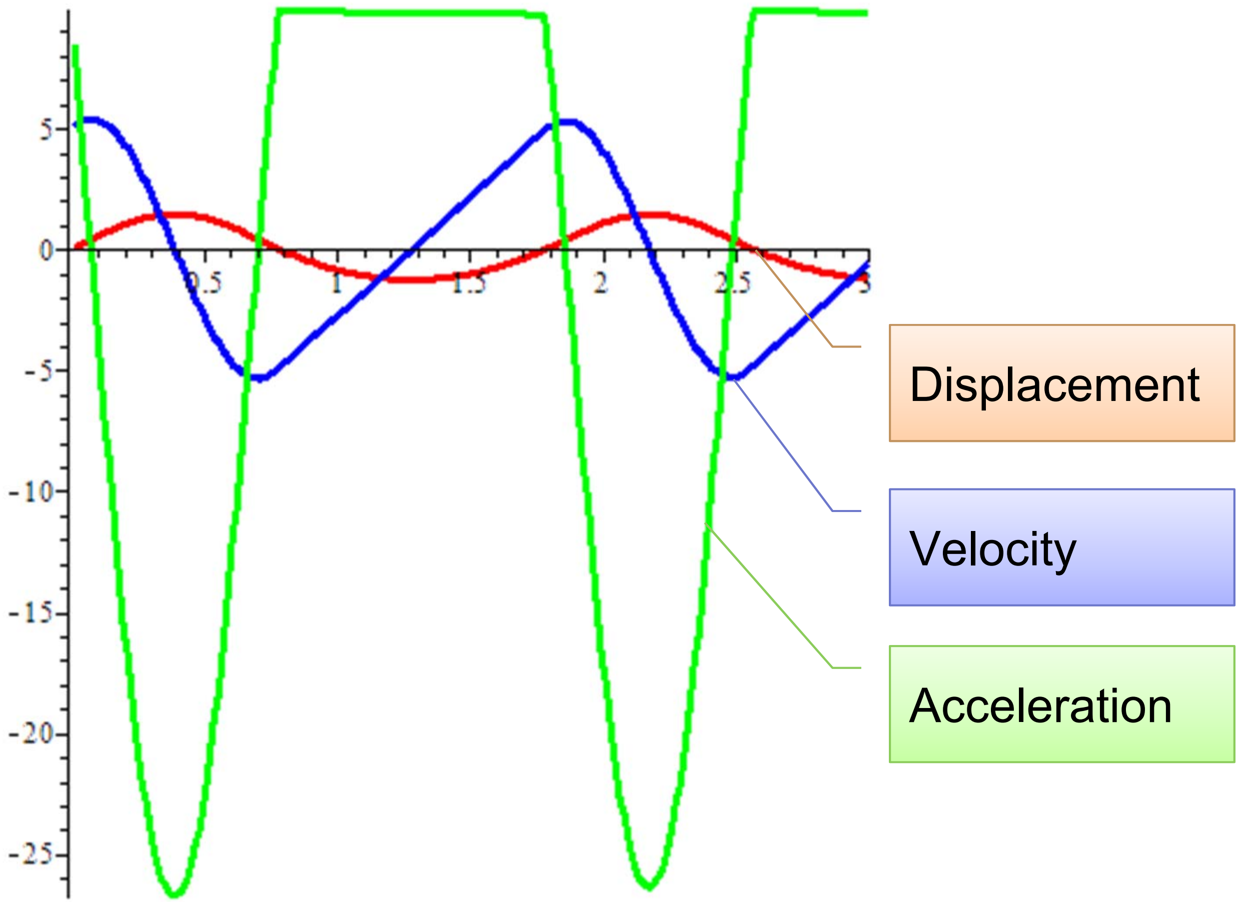
$$v := \mathbf{proc}(t) \ \ldots \ \mathbf{end\ proc}$$

acceleration

$> \ a := k \rightarrow fdiff(v(t), t = k);$

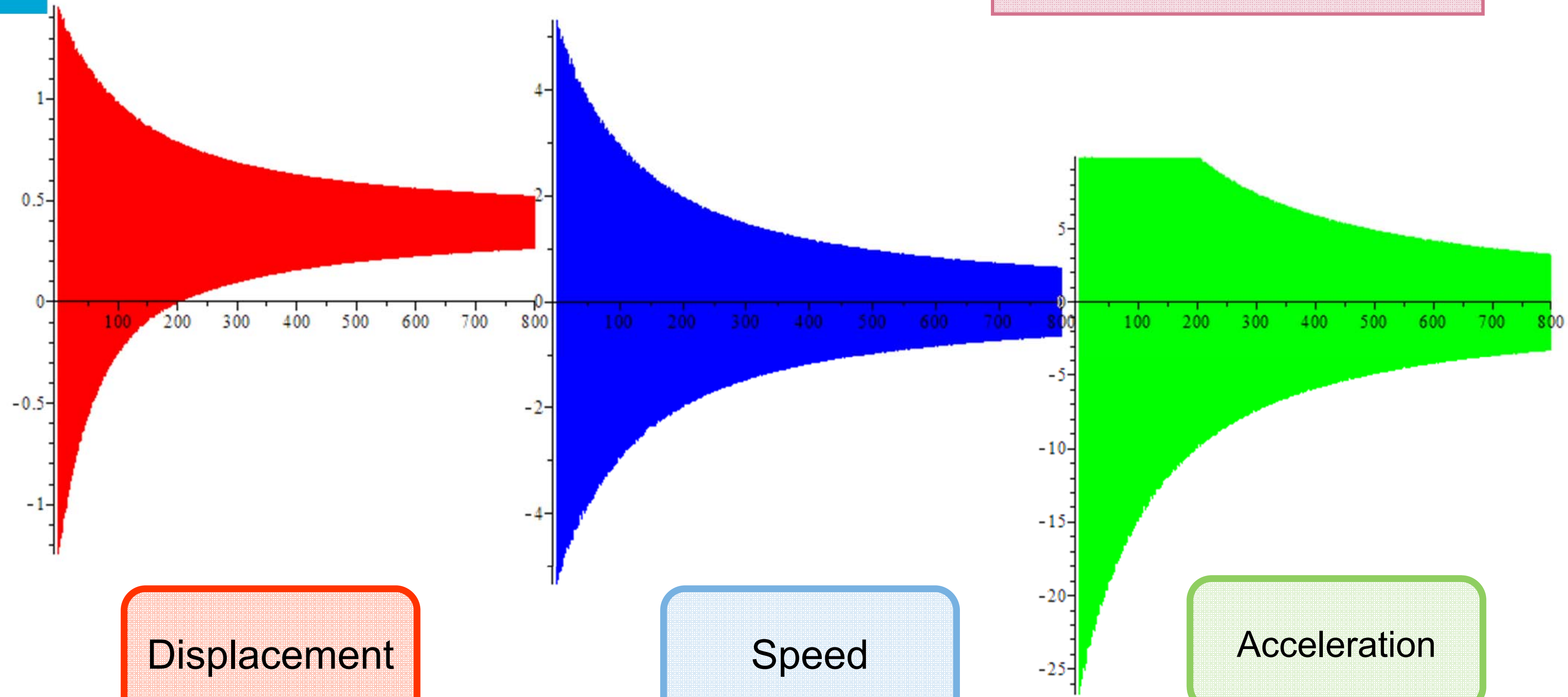
$$a := k \rightarrow fdiff(v(t), t = k)$$

The solution



The solution

The amplitude is also getting smaller. The same reason?



Displacement

Speed

Acceleration

Bungee jumping vs Trampoline

Bungee jumping

$$-m\frac{dx^2(t)}{dt^2} + mg - k\begin{cases} 0 & x(t) \leq L \\ x(t) - L & x(t) > L \end{cases} - signum(\frac{dx(t)}{dt})\frac{1}{2}\rho_{air}Ac_d(\frac{dx(t)}{dt})^2 = 0$$

$$-m\frac{dx^2(t)}{dt^2} + mg - k\begin{cases} 0 & x(t) < 0 \\ x(t) & x(t) \geq 0 \end{cases} - signum(\frac{dx(t)}{dt})\frac{1}{2}\rho_{air}Ac_d(\frac{dx(t)}{dt})^2 = 0$$

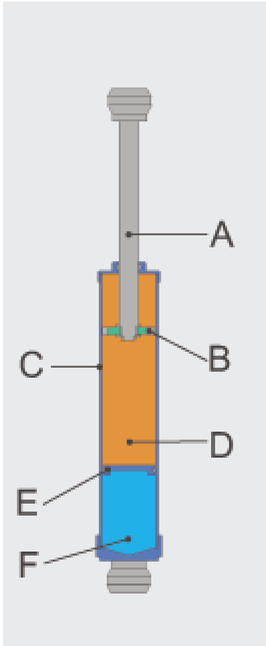
Trampoline

Components - Damper

Component - Damper

Damper

A damping elements resists relative velocity across them



Shock absorber with internal reservoir.

The components are:
A - rod,
B - the piston with seals,
C - the cylinder,
D - oil reservoir,
E - floating piston,
F - air chamber.

Courtesy of http://en.wikipedia.org/wiki/Shock_absorber

Model a damper

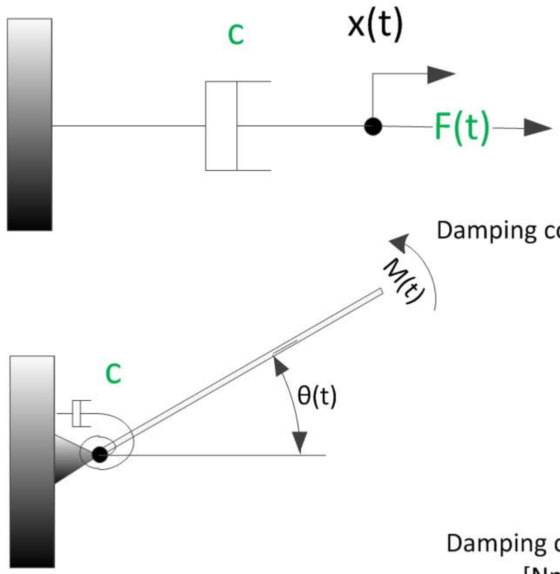
Damper – Linear

Courtesy of http://www.cartuningcentral.com/index.php?s=vibrations

Damper – Angular

Courtesy of http://www.topfreebiz.com/product/1893443/Toilet-Damper.htm

Sketch



Model

Force[N]

$$F(t) = -c\frac{dx(t)}{dt}$$

Damping coefficient [Ns/m]

Velocity [m/s]

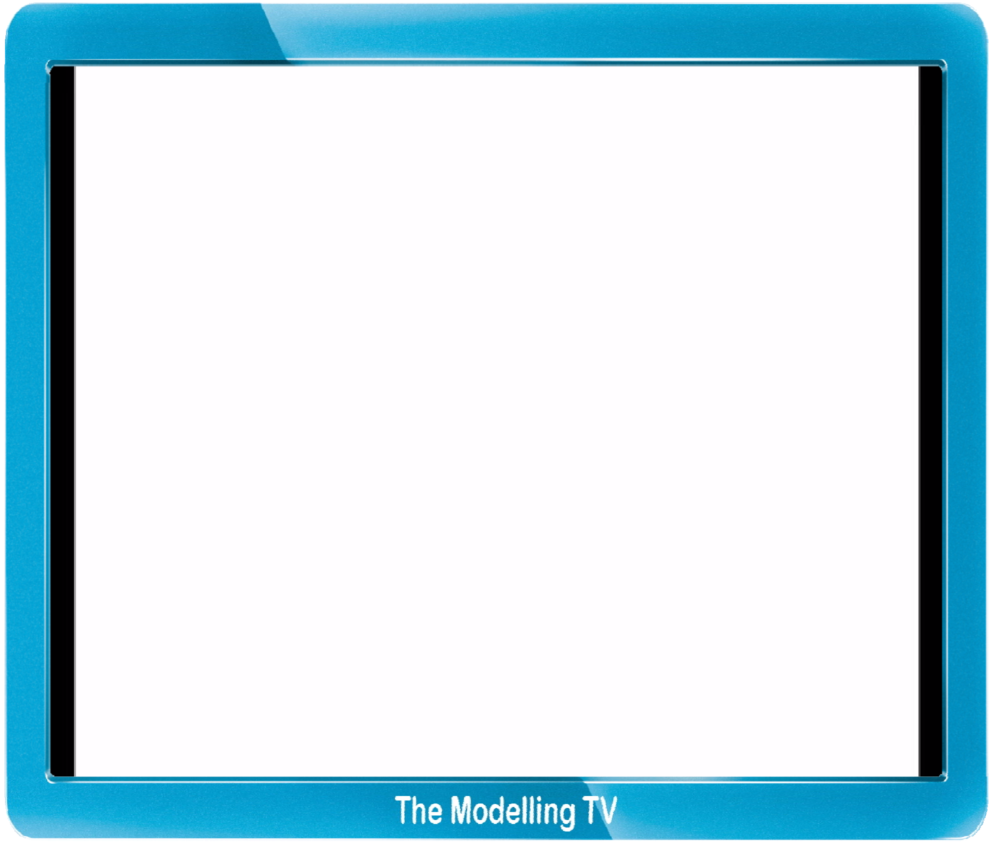
Torque[Nm]

$$M(t) = -c\frac{d\theta(t)}{dt}$$

Damping coefficient [Nms/radian]

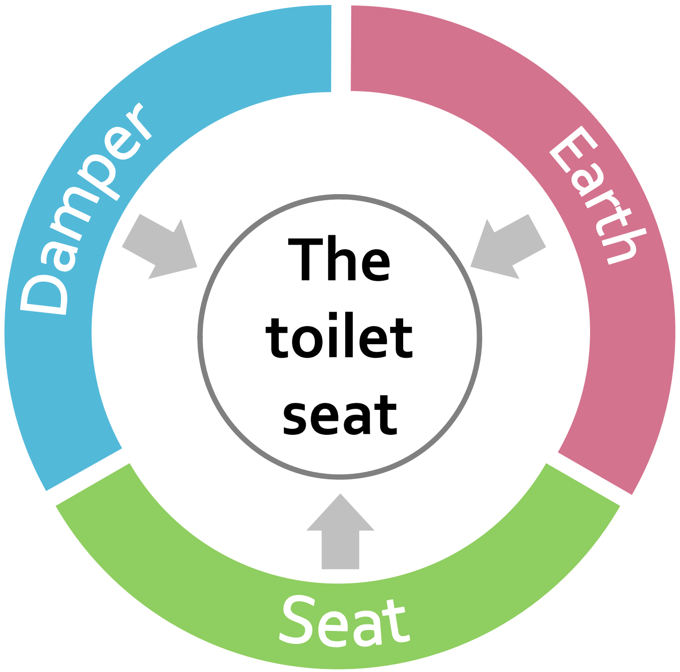
Angular Velocity [radian/s]

Case study: The slow closing toilet seat

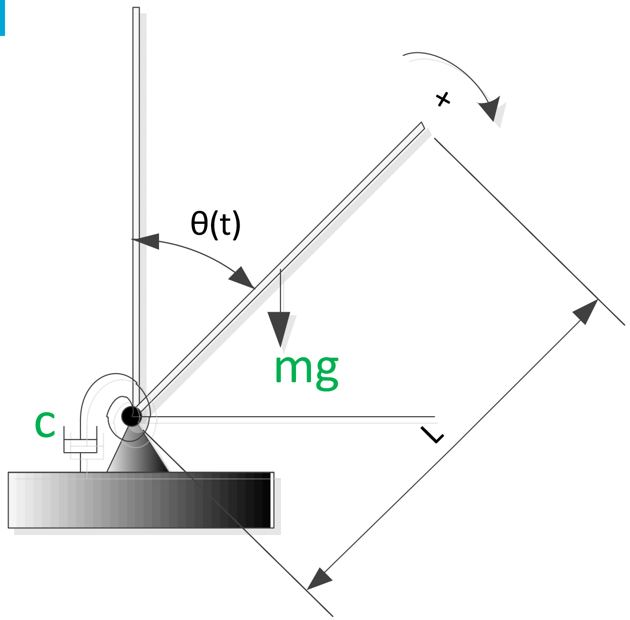


Courtesy of http://www.youtube.com/watch?v=tBK1rewI8vQ

System components



Cause-effect



Cause

Mass and geometry of the lid	Damper	Gravity

Effect

Inertia moment	Slow down the motion	The lid goes down

$$-I\frac{d\theta^2(t)}{dt^2} - C\left(\frac{d\theta(t)}{dt}\right) + mgsin(\theta(t))L/2=0$$

Modelling using Maple®

Toilet Seat

$\sum M = 0$

Inertia moment + Moment of damper + Moment of Gravity = 0
Suppose the angular diaplacement is theta(t)

> $restart:$

> $equ := -Inertia \cdot diff(\text{theta}(t), t\$2) - c \cdot diff(\text{theta}(t), t) + \frac{m \cdot g \cdot \sin(\text{theta}(t)) \cdot L}{2} = 0$

$$equ := -Inertia\left(\frac{d^2}{dt^2}\theta(t)\right) - c\left(\frac{d}{dt}\theta(t)\right) + \frac{1}{2}\, m\, g \sin(\theta(t))\, L = 0$$

Initial Conditions: 1 degree of angular displacement, zero angular velocity

> $ics := \text{theta}(0) = \frac{1 \cdot 3.14}{180}, D(\text{theta})(0) = 0;$

$$ics := \theta(0) = 0.01744444444, D(\theta)(0) = 0$$

Parameters (weight = 2kg, c=20Nm/s, arm length = 0.45 meters)

> $Inertia := m \cdot \left(\frac{L}{2}\right)^2 : g := 9.8 : L := 0.45 : m := 2 : c := 20 :$

Solving the model

Review

> *equu*

$$-0.1012500000\left(\frac{d^2}{dt^2}\theta(t)\right)-20\left(\frac{d}{dt}\theta(t)\right)+4.410000000\sin(\theta(t))=0$$

Solutions

> *sol* := *dsolve*({*equ*, *ics*}, theta(*t*), *numeric*, output = listprocedure, maxfun = 100000);

$$sol := \left[t = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc},\ \theta(t) = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc},\ \frac{d}{dt}\theta(t) = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc}\right]$$

displacement

> theta := *rhs*(*sol*[2]);

$$\theta := \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc}$$

angular velocity

> omega := *rhs*(*sol*[3]);

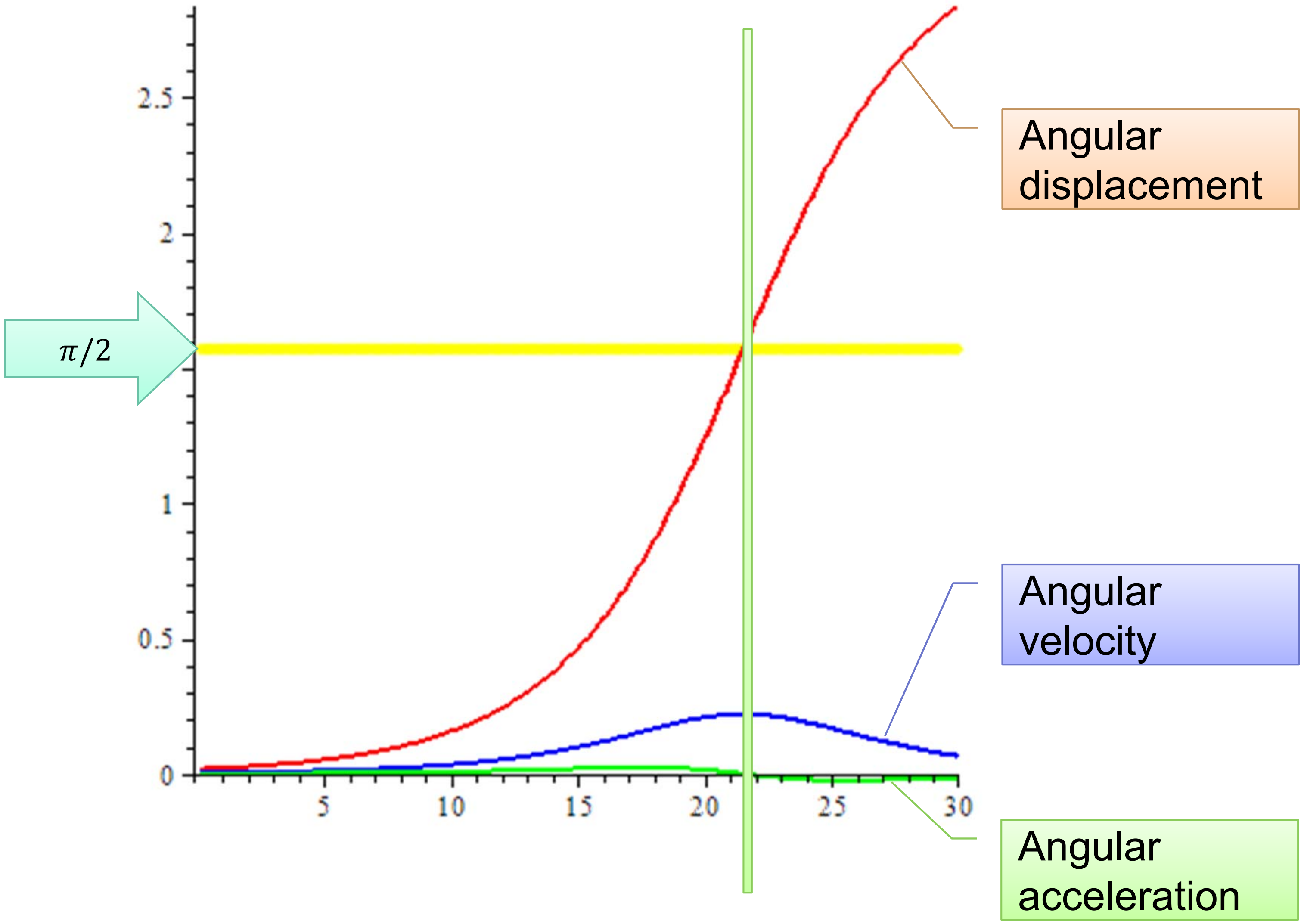
$$\omega := \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc}$$

angular acceleration

> alpha := *k*→*fdiff*(omega(*t*), *t* = *k*);

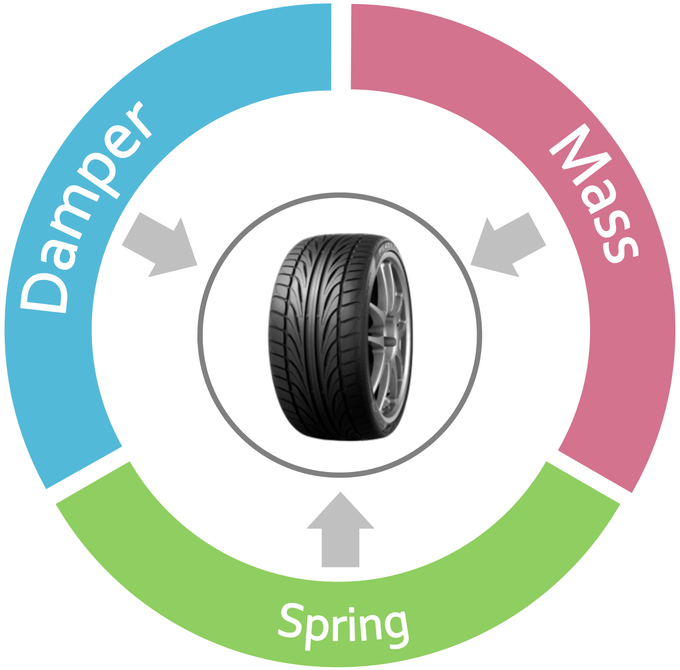
$$\alpha := k \rightarrow fdiff(\omega(t), t = k)$$

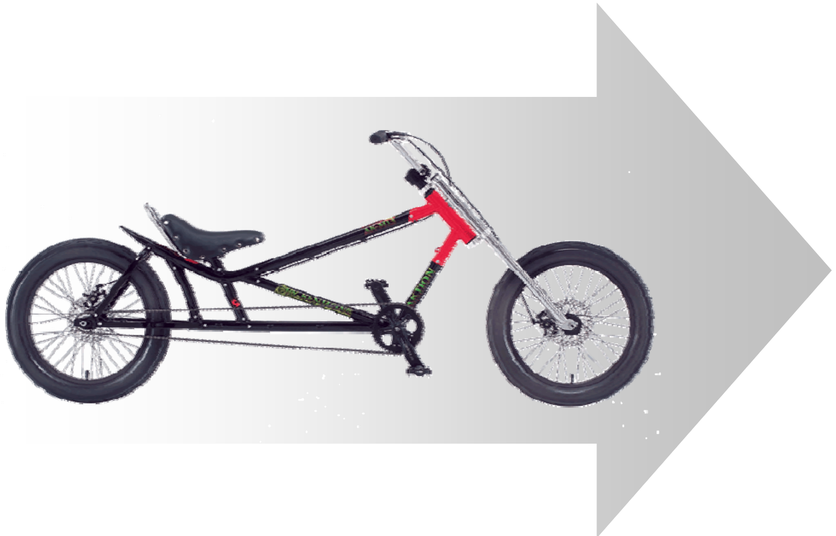
The solution



The mass-spring-damper system

The mass-spring-damper system





Courtesy of http://www.taiwan-bicycle.com.tw/Pd_Detail.asp?FKindNO=F000004&SKindNO=S000671&ItemNo=I00002896

The mass-spring-damper system

Regarding material

Spring

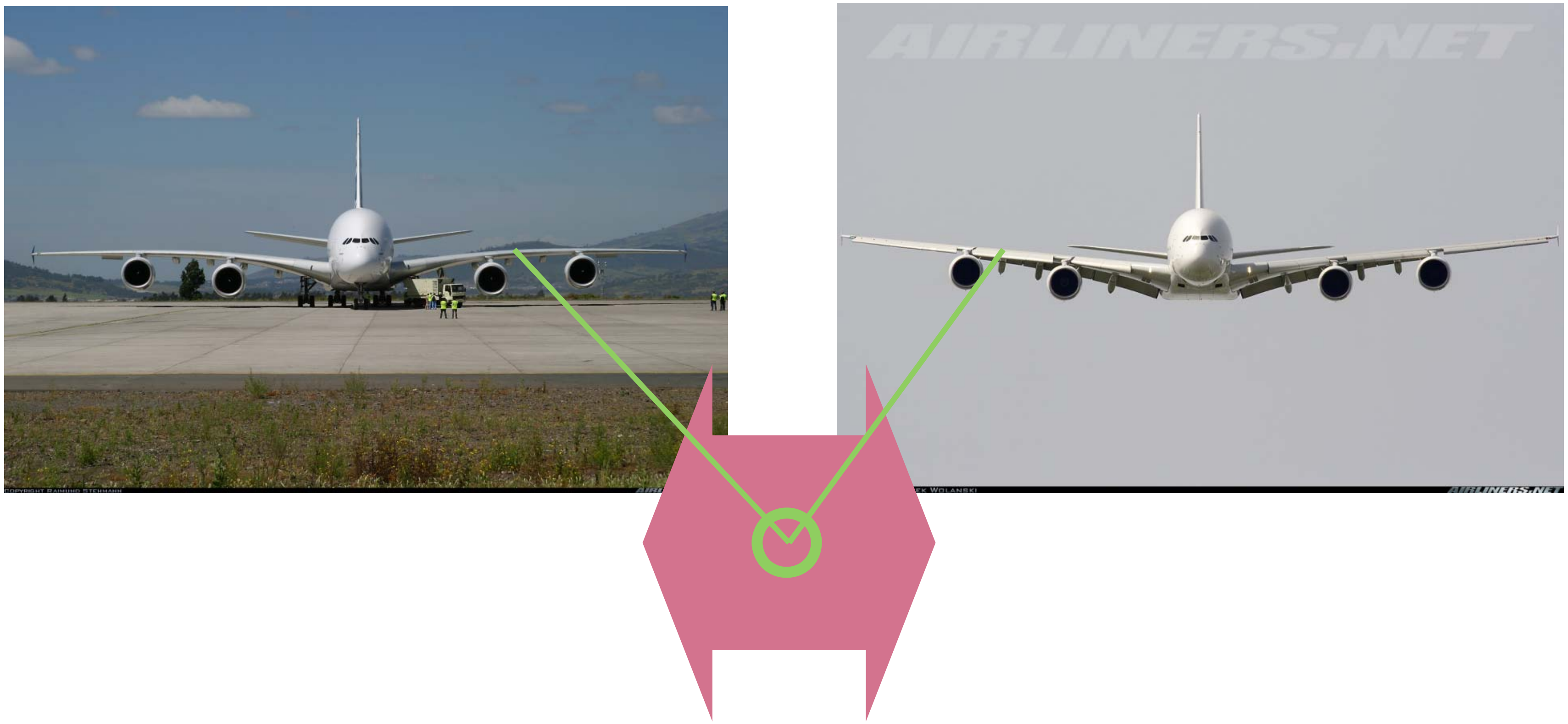
Elasticity
Young's Modulus
Poisson's Ratio
Anisotropy

Damper

Damping Coefficient

Courtesy of http://www.taiwan-bicycle.com.tw/Pd_Detail.asp?FKindNO=F000004&SKindNO=S000671&ItemNo=I00002896

The mass-spring-damper system



Courtesy http://www.airliners.net/photo/Airbus-Industrie/Airbus-A380-861/1169635/L/

Case study: The skateboard
The movement in the vertical direction

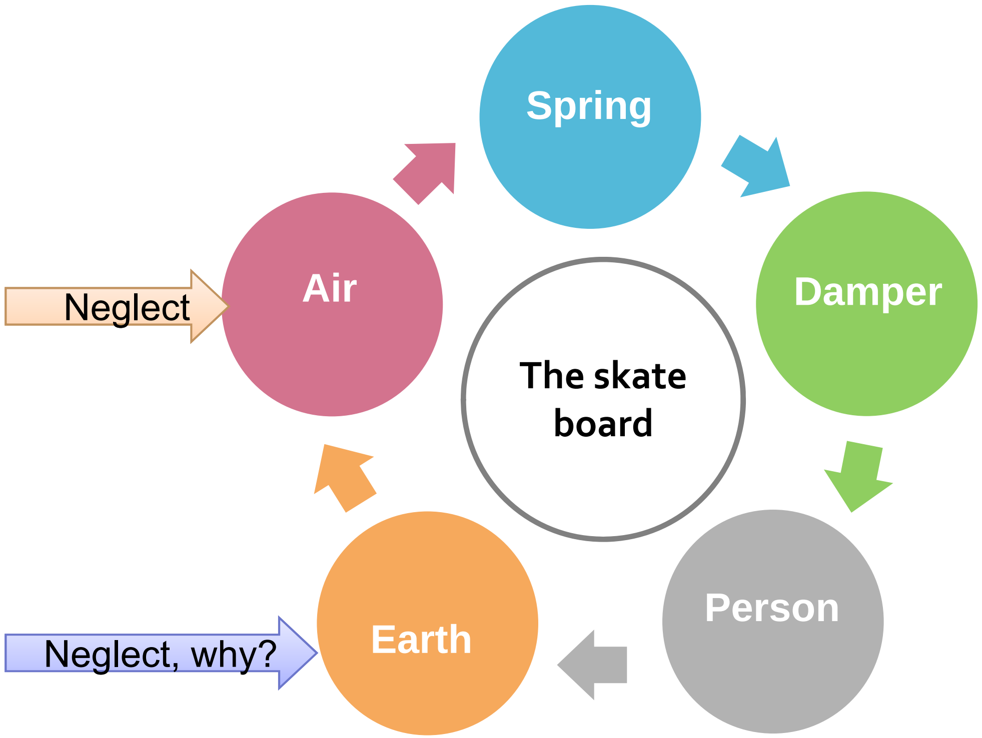
The World largest skateboard

Designer:

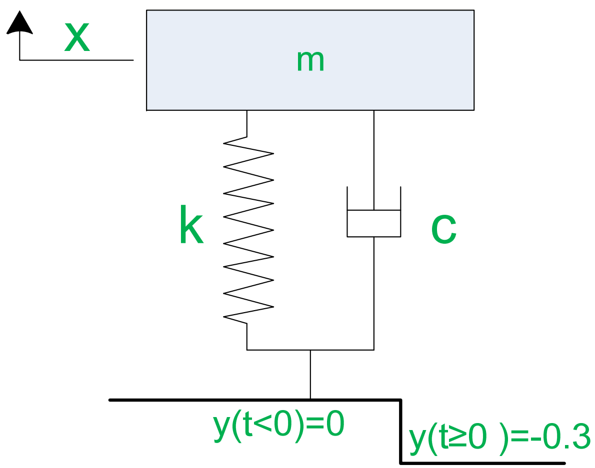
Rob Dyrde

Courtesy of http://skateboardingmagazine.com/blog/2009/02/01/the-worlds-largest-skateboard/

System components



System



Cause

Mass of the board	Damper	Spring

Effect

Inertia force	Slow down the motion	Drag the mass back

$$-m\frac{dx^2(t)}{dt^2} \quad - c\left(\frac{dx(t)}{dt}-\frac{dy(t)}{dt}\right) - k\big(x(t)-y(t)\big)=0$$

Modelling using Maple®

Mass Spring Damper - Step

$\sum F = 0$

Inertia force + Force of Spring + Force of damper= 0
Suppose the displacement is x(t)

> *restart* :

> $equ := -m \cdot diff(x(t), t\$2) - c \cdot (diff(x(t), t) - diff(y(t), t)) - k \cdot (x(t) - y(t)) = 0$

$$equ := -m\left(\frac{d^2}{dt^2}x(t)\right) - c\left(\frac{d}{dt}x(t) - \left(\frac{d}{dt}y(t)\right)\right) - k(x(t) - y(t)) = 0$$

Parameters

> $m := 50 : g := 9.8 : c := 100 : k := 1000 : y := t \rightarrow -0.3 \cdot \mathrm{Heaviside}(t) :$

$$y := t \rightarrow (-1) \cdot 0.3\,\mathrm{Heaviside}(t)$$

Review

> *equ*

$$-50\left(\frac{d^2}{dt^2}x(t)\right) - 100\left(\frac{d}{dt}x(t)\right) - 30.0\,\mathrm{Dirac}(t) - 1000\,x(t) - 300.0\,\mathrm{Heaviside}(t) = 0$$

Initial conditions

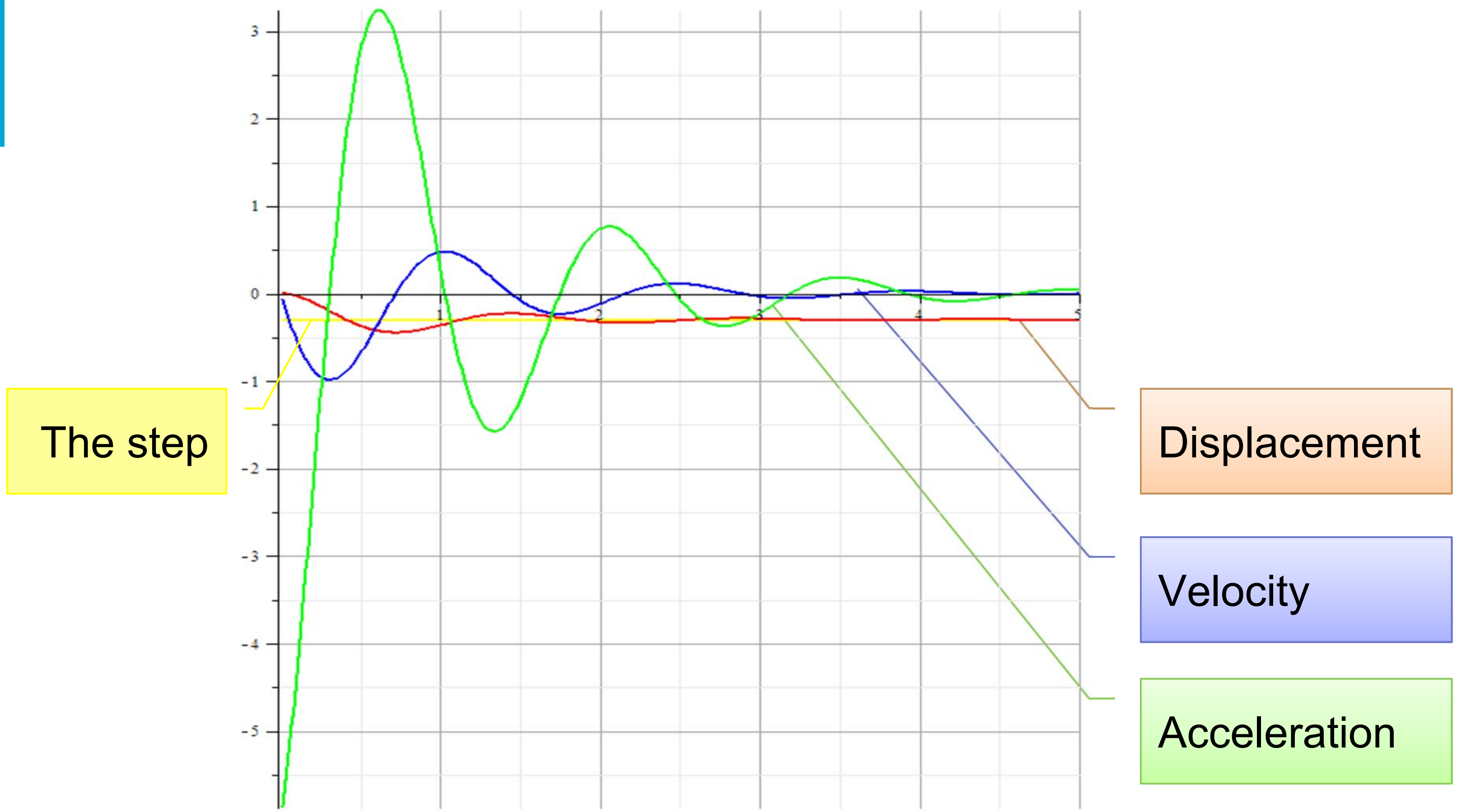
> $ics := x(0) = 0, \mathrm{D}(x)(0) = 0 :$

Solutions

> $sol := dsolve(\{equ, ics\}, x(t), numeric, output = listprocedure)$

$$sol := \left[t = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc}, x(t) = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc}, \frac{d}{dt}x(t) = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc}\right]$$

Solution



3
2
1
0
-1
-2
-3
-4
-5
1
2
3
4
5
The step
Displacement
Velocity
Acceleration

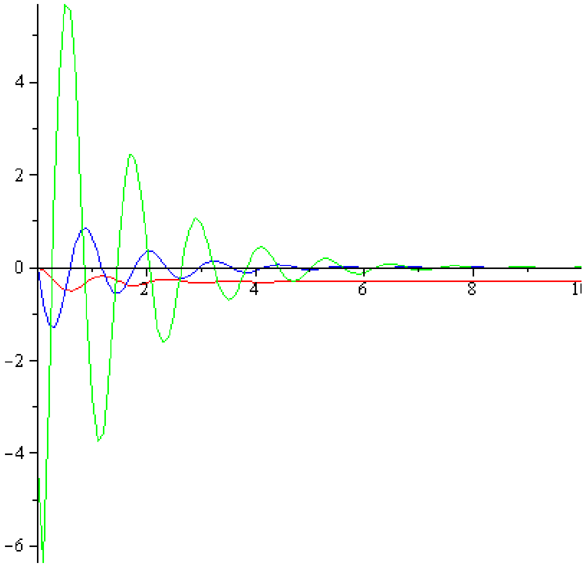
Discussion

C=50

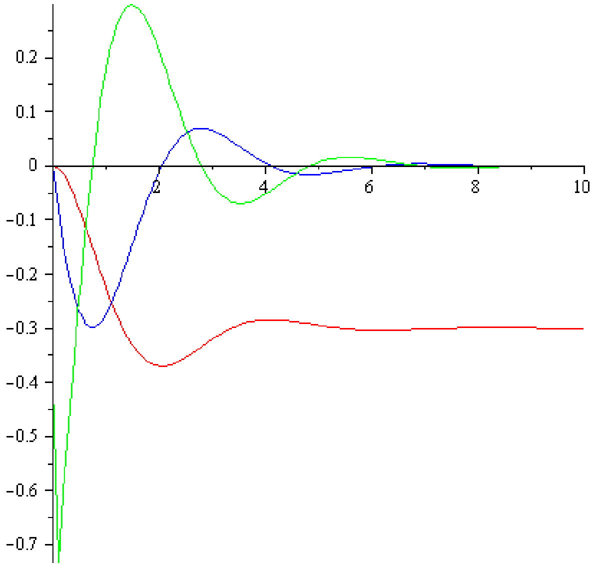
Displacement

Velocity

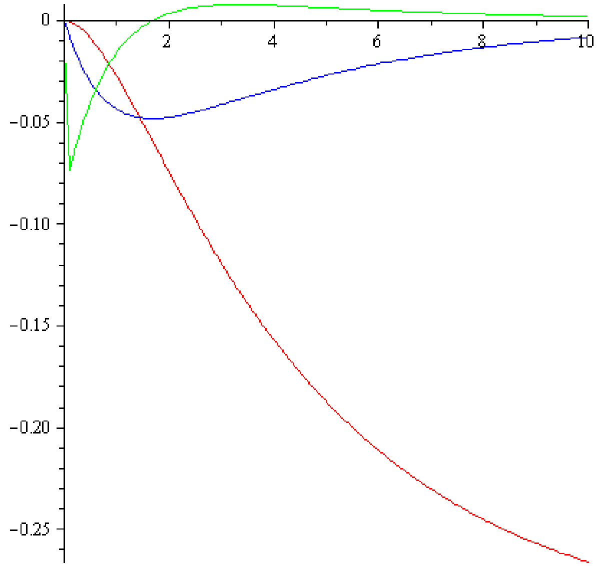
Acceleration



K=1000



K=100



K=10

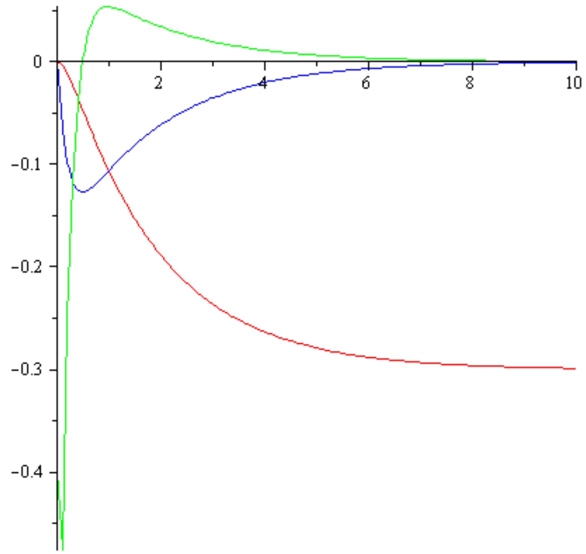
Discussion

K=100

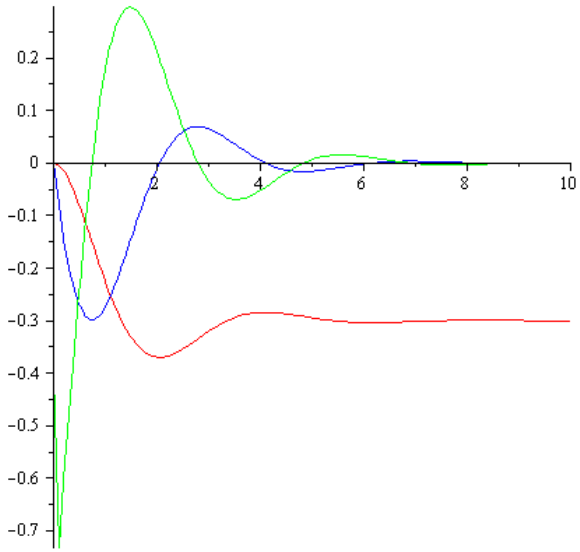
Displacement

Velocity

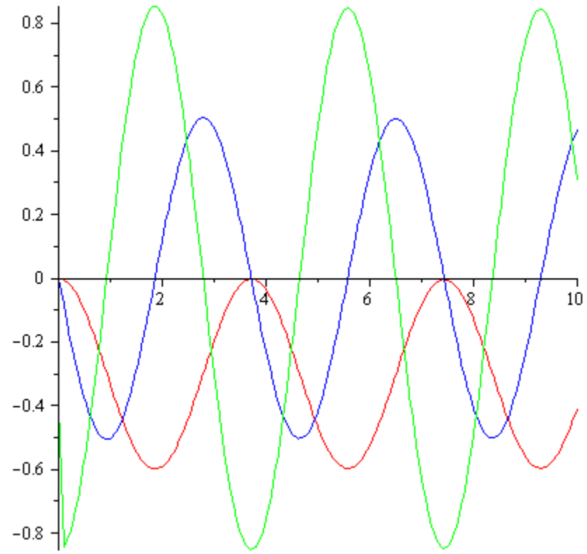
Acceleration



C=200

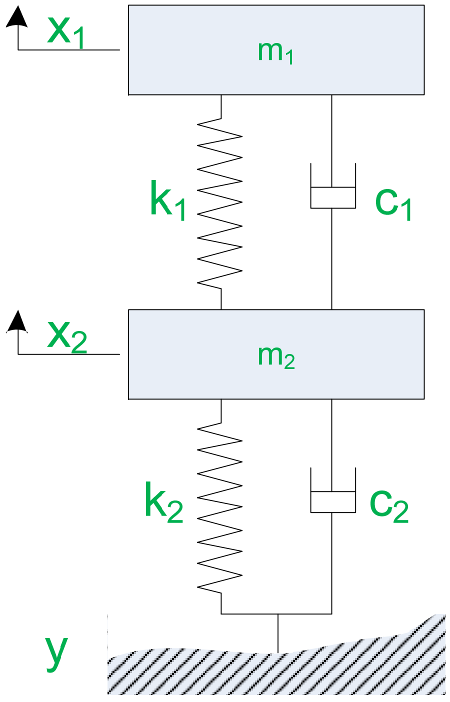


C=50



C=0.1

Consider human also as a spring and a damper, can we model the shock to the head?



We suppose the mass (mr+mb) moves slightly in the X1 + direction			
Cause	Mass of bike and the human	Spring ks	Damper cs
Effect	Inertia force in x-	Drag the mass in x- direction	Drag the mass (motion) back in x- direction

Consider all forces along the vertical direction

$$-m_1\frac{d^2x_1(t)}{dt^2}-k_1(x_1(t)-x_2(t))-c_1(\frac{dx_1(t)}{dt}-\frac{dx_2(t)}{dt})=0$$

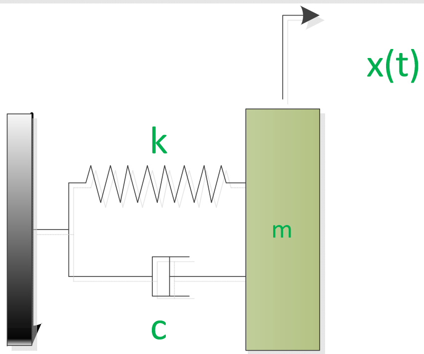
We suppose the mass (mw) moves slightly in the X_2 + direction					
Cause	Mass of bike and the human	Reaction force from Spring k_s	Reaction force from Damper c_t	Spring k_t	Damper c_t
Effect	Inertia force in X_2-	Pull the mass in X_2+ direction	Slow down the motion of the mass in X_2+ direction	Drag the mass in X_2- direction	Slow down the motion of the mass in X_2- direction

Consider all forces along the vertical direction

$$-m_2\frac{d^2x_2(t)}{dt^2}+k_1(x_1(t)-x_2(t))+c_1(\frac{dx_1(t)}{dt}-\frac{dx_2(t)}{dt})-k_2(x_2(t)-y(t))-c_2(\frac{dx_2(t)}{dt}-\frac{dy(t)}{dt})=0$$

The natural frequency

Given a mass-spring damper system



It can be modeled as:

$$-m\frac{d^2x(t)}{dt^2}-c\frac{dx(t)}{dt}-kx(t)=0$$

Introduce two parameters

$\omega_0=\sqrt{\frac{k}{m}}$	Name: Natural frequency	Unit: Radians/Second
$\zeta=\frac{c}{2\sqrt{mk}}$	Name: Damping Ratio	Unit: dimensionless

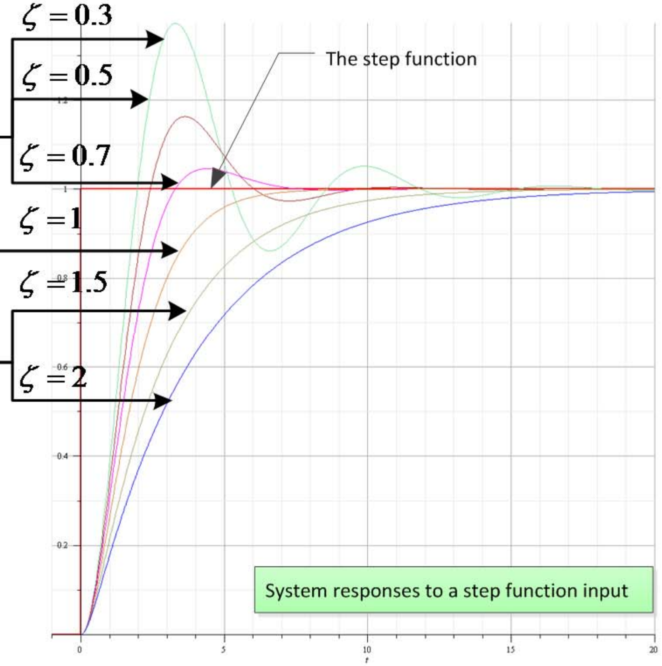
The model turns to:

$$\frac{d^2x(t)}{dt^2}+2\zeta\omega_0\frac{dx(t)}{dt}+\omega_0^2x(t)=0$$

The damping ratio

$$\zeta = \frac{c}{2\sqrt{mk}}$$

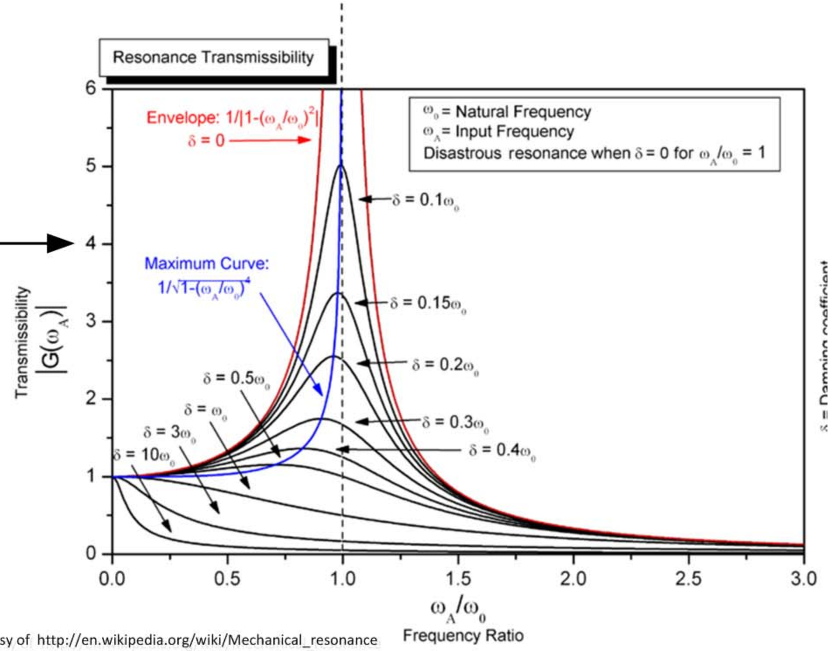
$0 \le \zeta < 1$	Under-damping	Solution is a complex, system will oscillate at natural frequency
$\zeta = 1$	Critical-damping	System converges to the input in the fastest way
$\zeta > 1$	Over-damping	System converges to the input longer than critical damping, but it is more stable



Resonance

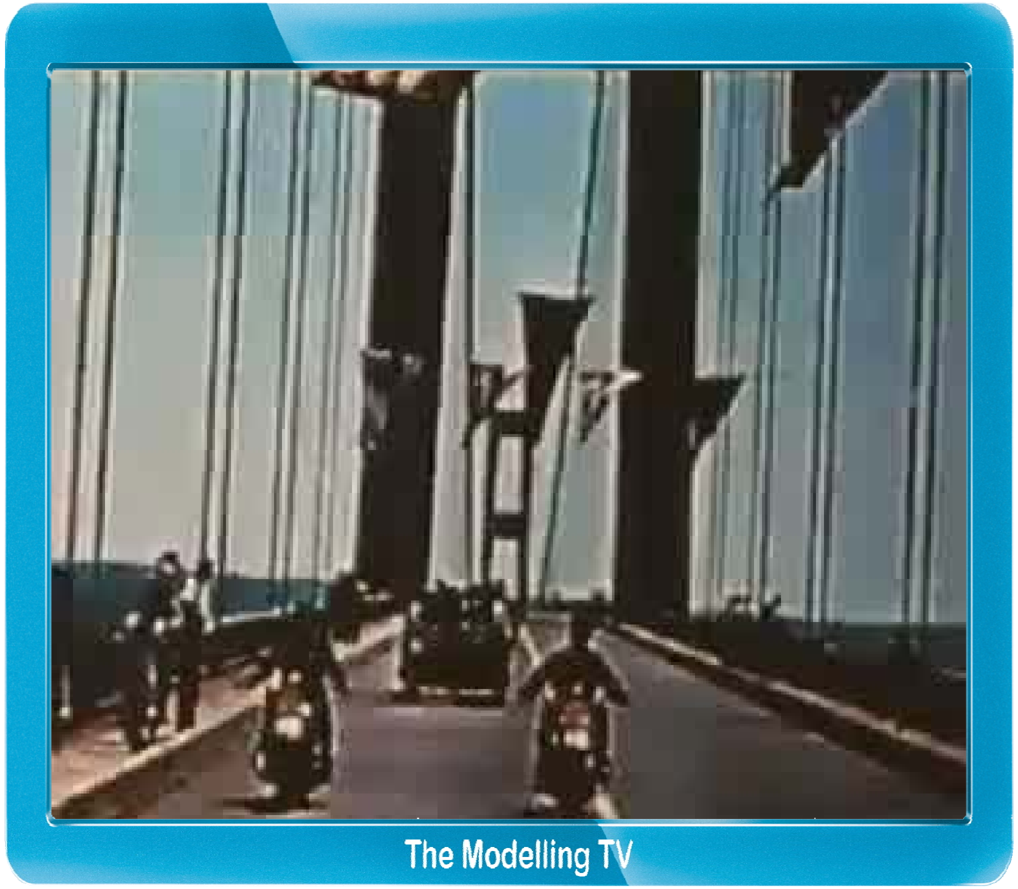
$$\omega_0 = \sqrt{\frac{k}{m}}$$

If the frequency of the input is the same as the natural frequency, and the damping ratio is less than 1, it will lead to resonance.



Courtesy of http://en.wikipedia.org/wiki/Mechanical_resonance

Case study: Tacoma Narrow bridge



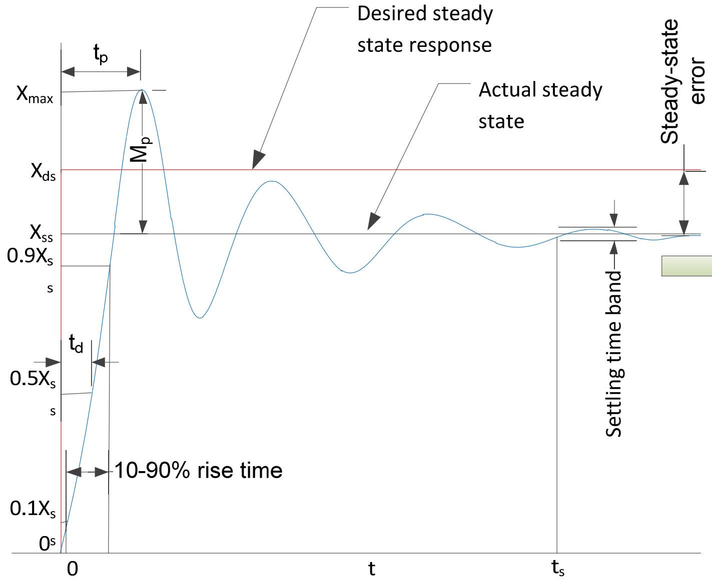
Courtesy of http://www.youtube.com/watch?v=P0Fi1VcbpAI

Interpret the response of a step function

System

$$\frac{d^2x(t)}{dt}+2\zeta\omega_0\frac{dx(t)}{dt}+\omega_0^2x(t)=f_{step}$$

Suppose we got such a response regarding a step function



Ref. William Palm III, System Dynamics, McGraw-Hill Science/Engineering/Math; 2 edition, January 26, 2009

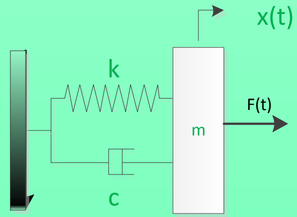
Maximum overshoot ratio $A=\frac{x_{ss}}{x_{max}-x_{ss}}$

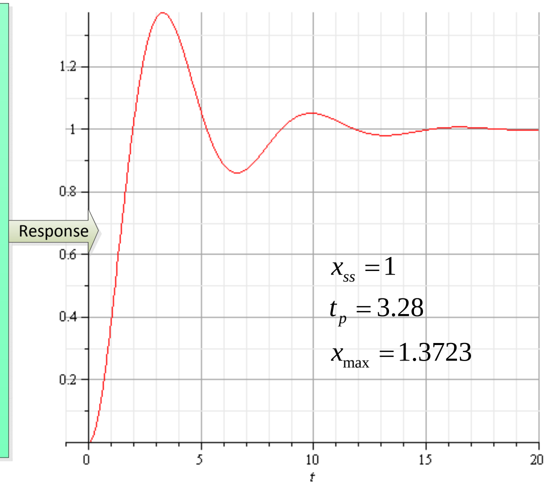
Damping ratio $\zeta=\frac{\ln A}{\sqrt{\pi^2+(\ln A)^2}}$

Natural frequency $\omega_n=\frac{\pi}{t_p\sqrt{1-\zeta^2}}$

Interpret the response - 2

Example: If we know the response of a mass-spring-damper system regarding a step function F(t), can we extract parameter c and k from the response?





Maximum overshoot ratio $A = \dfrac{x_{ss}}{x_{max} - x_{ss}}$

Maximum overshoot ratio $A = 2.686$

Damping ratio $\zeta = \dfrac{\ln A}{\sqrt{\pi^2 + (\ln A)^2}}$

Damping ratio $\zeta = 0.3$

Natural frequency $\omega_n = \dfrac{\pi}{t_p\sqrt{1-\zeta^2}}$

Natural frequency $\omega_n = 1$

From definitions $\omega_0 = \sqrt{\dfrac{k}{m}}$ $\zeta = \dfrac{c}{2\sqrt{mk}}$

Suppose m=1kg $k = 1 \text{ kg/m}, c = 0.6 \text{ Ns/m}$

Industrial applications

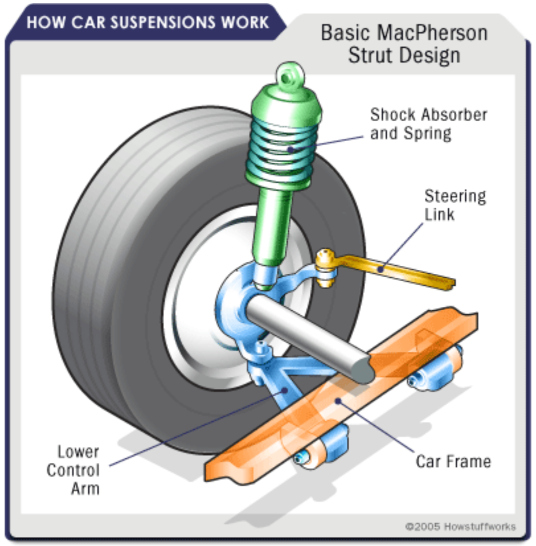
The pogo-stick

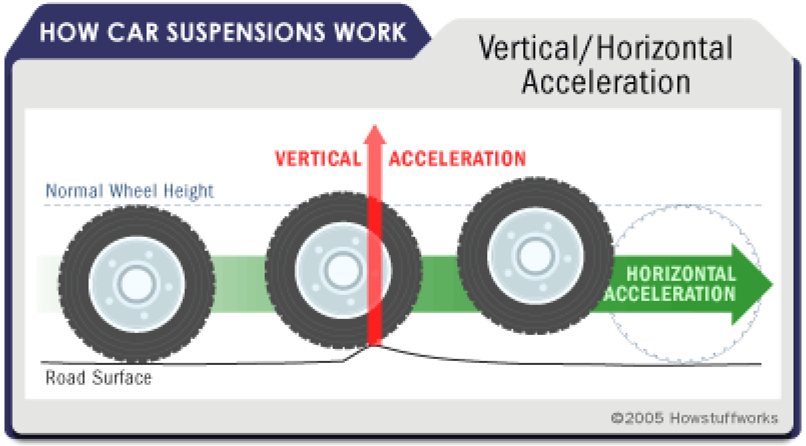
Question?

Where is the damper?

Courtesy of http://connect.in.com/pogo/photos-1-1-1-9ed7a7cecf4cb85de07b7f05697bfe43.html

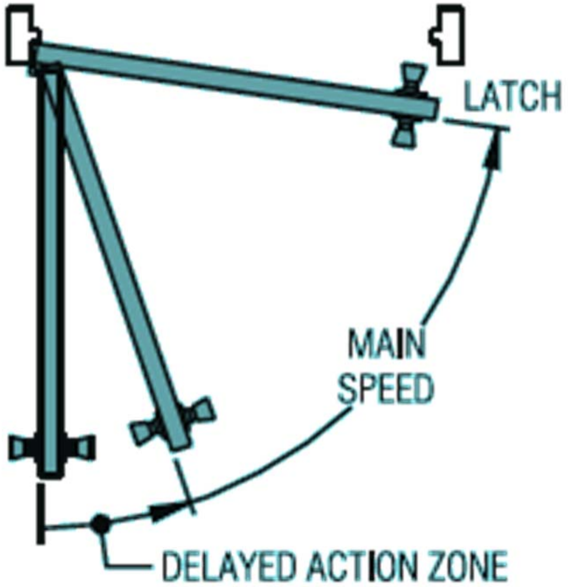
The suspension systems (ref/auto.howstuffworks.com/)

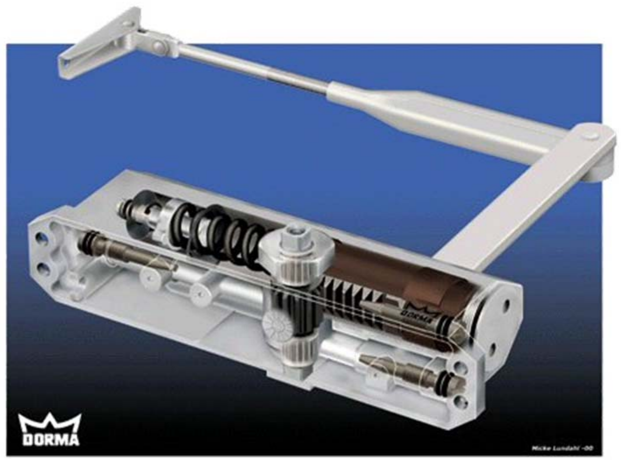




Courtesy of http://auto.howstuffworks.com/car-suspension4.htmAl

The door closer (ref. /www.usbuildersupply.com)





Courtesy of http://www.usbuildersupply.com/index.php/hardware-basics/door-closer-overview/

Thank You!