Pen and Paper Exercises - linear combinations and linear independence

1. Show that $\mathbb{R}^2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

2. Show that $\mathbb{R}^3 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

3. Describe the span of the vectors both geometrically and algebraically.

   (a) $\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \end{bmatrix} \right\}$

   (b) $\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}$

4. The subspace $H = \text{Span}\,\{\mathbf{u},\ \mathbf{u}+2\mathbf{v},\ \mathbf{u}-\mathbf{v}+3\mathbf{w}\}$ is given, where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

   (a) Show that $\mathbf{u} \in H$.

   (b) Show that $\mathbf{v} \in H$.

   (c) Show that $\mathbf{w} \in H$.

   (d) Show that $\mathbf{u}+\mathbf{v}+\mathbf{w} \in H$.

5. Prove the following statements using the the relevant definitions, or disprove the statement using an appropriate counterexample.

   (a) If $S = \{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is linearly independent, then $T = \{\mathbf{u}+\mathbf{v},\ \mathbf{v}+\mathbf{w},\ \mathbf{u}+\mathbf{w}\}$ is linearly independent.

   (b) If $S = \{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is linearly independent, then $T = \{\mathbf{u}-\mathbf{v},\ \mathbf{v}-\mathbf{w},\ \mathbf{u}-\mathbf{w}\}$ is linearly independent.

   (c) Every subset of a linearly independent set is linearly independent.

   (d) Every subset of a linearly dependent set is linearly dependent.

   (e) A set of two vectors is linearly independent if and only if one of the vectors is a multiple of the other vector.