Pen and Paper Exercises - matrix operations

1. The following linear system is given:

$$\begin{cases} x_1 + 2x_2 + 3x_3 &= 4 \\ 5x_1 + 6x_2 + 7x_3 &= 8 \end{cases} \tag{1}$$

   (a) Rewrite system (1) as a vector equation and explain why your vector equation is equivalent to the system.

   (b) Use the definiton of the matrixvector product to rewrite your vector equation as a matrix equation $A\mathbf{x} = \mathbf{b}$.

2. The matrix A and the vector $\mathbf{b}$ are given. Write $A\mathbf{b}$ as a linear combination of the columns of A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}. \tag{2}$$

3. Prove the following statements using the the relevant definitions, or disprove the statement using an appropriate counterexample.

   (a) If the columns of a matrix B are independent, then the columns of AB are independent.

   (b) If the columns of a matrix B are dependent, then the columns of AB are dependent.

   (c) If A and B are square matrices and if AB is invertible, then A is invertible.

4. Does the equation $(A+B)(A-B) = A^2 - B^2$ hold for all square matrices A and B?

5. The matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given. Find conditions on a, b, c and d such that $AB = BA$, where

   (a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

6. Solve the given matrix equation for X and simplify as much as possible (there should be no brackets in your final answer); A, B and C are $n \times n$ matrices.

   (a) $ABXA^{-1}B^{-1} = I_n + A$

   (b) $AXB - BCA = CB$

   (c) $AXB = (BA)^2$