Pen and Paper Exercises - subspaces, basis and dimension

1. Definition: a subset H of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if it satisfies the following three properties:

   A $\mathbf{0} \in H$ (H contains the zero vector)

   B For all $\mathbf{u}, \mathbf{v} \in H, \mathbf{u} + \mathbf{v} \in H$ (closed under addition)

   C For all $\mathbf{u} \in H$ and for all scalars $c \in \mathbb{R}, c\mathbf{u} \in H$ (closed under scalar multiplication)

   (a) The subset $H_1 = \text{Span}\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\} \subset \mathbb{R}^2$ is given. Show that H_1 satisfies all three properties of a subspace (i.e. H_1 is a subspace of $\mathbb{R}^2$).

   (b) The subset $H_2 = \left\{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\right\} \subset \mathbb{R}^2$ is given.

      i. Show that properties A and B are satisfied.

      ii. Try to show that property C is satisfied. Why isn't this property satisfied?

      iii. Give an explicit counterexample of property C (this means that H_2 is not a subspace of $\mathbb{R}^2$).

   (c) The subset $H_3 = \left\{(x, y) \in \mathbb{R}^2 | xy \geq 0\right\} \subset \mathbb{R}^2$ is given.

      i. Show that properties A and C are satisfied.

      ii. Try to show that property B is satisfied. Why isn't this property satisfied?

      iii. Give an explicit counterexample of property B (this means that H_3 is not a subspace of $\mathbb{R}^2$).

2. It is given that A is a 2×4 matrix.

   (a) True or false (motivate your answer): the columns of A dependent.

   (b) Give all possible values of the dimension of Nul A. Motivate your answer.

3. Determine dim Col A for all values of h, where $A = \begin{bmatrix} h & 1 & 2 \\ 0 & h-1 & 0 \\ 0 & 0 & h^2-1 \end{bmatrix}$.