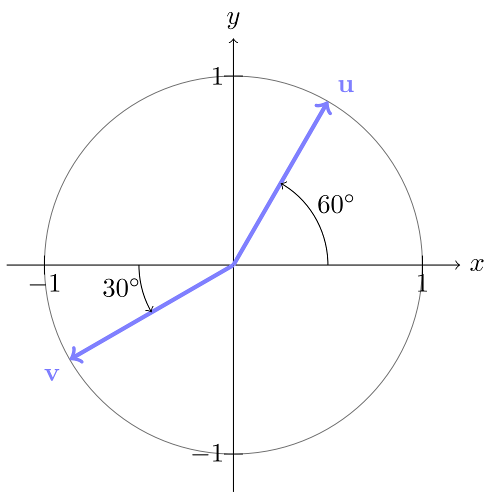
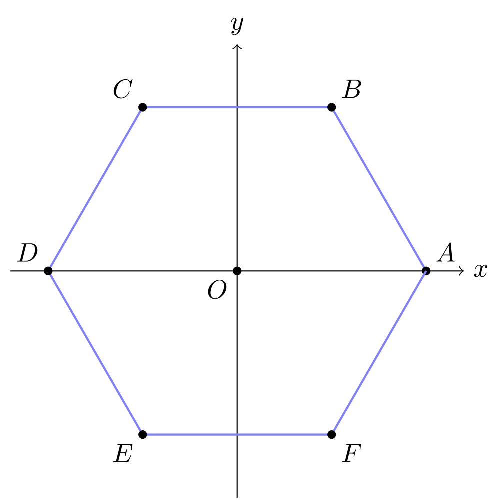
Pen and Paper Exercises - vectors, lines and planes

1. For each of the following pair of points A and B, draw the vector $\mathbf{v}$ from A to B. Then compute $\mathbf{v}$ and redraw it as a vector in standard position.
   (a) A: (2,3) B: (4,5)
   (b) A: (-1,3) B: (-3, -1)

2. (a) Find the components of the vectors $\mathbf{u}$, $\mathbf{v}$, which are shown in the figure below.
   (b) Draw vectors $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ and find their components.



3. In the figure below, A, B, C, D, E and F are the vertices of a regular hexagon entered at the origin. Express each of the following vectors in terms of $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$.
   (a) $\overrightarrow{AB}$
   (b) $\overrightarrow{AD}$
   (c) $\overrightarrow{BC}$
   (d) $\overrightarrow{CF}$
   (e) $\overrightarrow{AC}$
   (f) $\overrightarrow{BC} + \overrightarrow{DE} + \overrightarrow{FA}$



4. $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbf{R}^n$ and c is a scalar. Are the following expressions well defined or not? Explain your reasoning.

   (a) $||\mathbf{u} \cdot \mathbf{v}||$

   (b) $|\mathbf{u} \cdot \mathbf{v}|$

   (c) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

   (d) $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$

   (e) $(\mathbf{u} \cdot \mathbf{v}) + \mathbf{w}$

   (f) $c \cdot (\mathbf{u} + \mathbf{w})$

   (g) $c(\mathbf{u} + \mathbf{w})$

   (h) $c(\mathbf{u} \cdot \mathbf{w})$

5. Describe the set of all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to the vector $\mathbf{u}$, where

   (a) $\mathbf{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

   (b) $\mathbf{u} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

6. Prove the following statements using the the relevant definitions, or disprove the statement using an appropriate counterexample.

   (a) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$.

   (b) $||\mathbf{u} - \mathbf{v}|| \geq |||\mathbf{u}|| - ||\mathbf{v}|||$ (the reverse triangle inequality).

   (c) If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ then $\mathbf{v} = \mathbf{w}$.

   (d) If $\mathbf{u}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, then $\mathbf{u}$ is orthogonal to all vector $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}$, where c_1, c_2 are arbitrary scalars.

   (e) If $\mathbf{u}$ is orthogonal to all vector $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}$, where c_1, c_2 are arbitrary scalars, then $\mathbf{u}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$.

7. Prove the following properties of the cross product ($\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$).

   (a) $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$

   (b) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

   (c) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

   (d) $||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2$

   (e) $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.