Chapter 10: Stress Tensors

STRESS IS a force per unit area with the mathematical properties of a tensor quantity. In simple applications, stress can be discussed in terms of principal stresses. Such principal stresses can be represented by a stress ellipsoid, as outlined in chapters three and four, without resorting to the tensor notation. However, for the description of complex stress fields, tensor notation is much more powerful than the ellipsoid equation. The fundamental principles of tensor calculus and its practical application to stress problems are outlined. Stress tensors should be handled with care, preferably applying them only to those situations which require a sophisticated notation.

Contents: This chapter first explains the limitations of the ellipsoid description of stress and introduces the stress tensor (section 10-1). Subsequently, the mathematical concepts of direction cosines and vectorial index notation are introduced (section 10-2), to allow the description of normal and shear stress on an arbitrary plane (section 10-3). The elements of the stress tensor are defined (section 10-4), followed by a short explanation on the importance of choosing an appropriate coordinate system for each application in order to obtain stress tensors with simple elements (section 10-5). Complementary mathematical concepts are outlined to facilitate the manipulation of stress tensors: index notation for tensors (section 10-6), Cauchy's formula and the summation convention (section 10-7), the cubic equation (section 10-8), and the tensor transformation formula (section 10-9). The chapter concludes with an outline of the special case of plane stress in 2-D (section 10-10) and summarizes the significance of the stress tensor invariants (section 10-11).

Practical hint: **Stress can be visualized in photo-elastic media using polarized light. Examples of photo-elastics are gelatin and plexiglass. Geological applications have been published by Arvid Johnson (1977, *Styles of Folding*, Elsevier). Design your own photo-elastic experiment.**

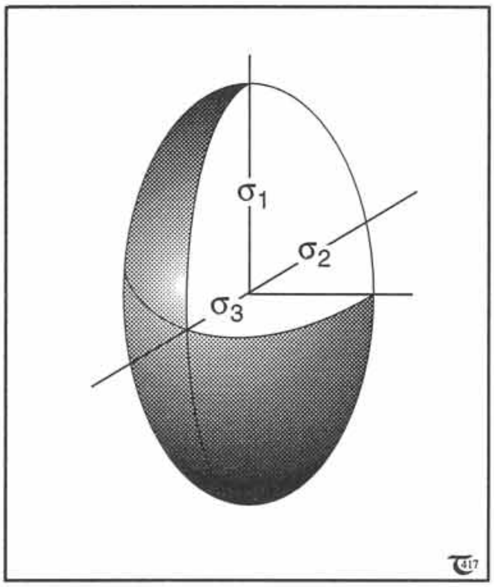

Figure 10-1: State of stress in an infinitesimal point can be represented by a finite stress-ellipsoid, with axial lengths determined by the three principal stresses, σ_1, σ_2, and σ_3.

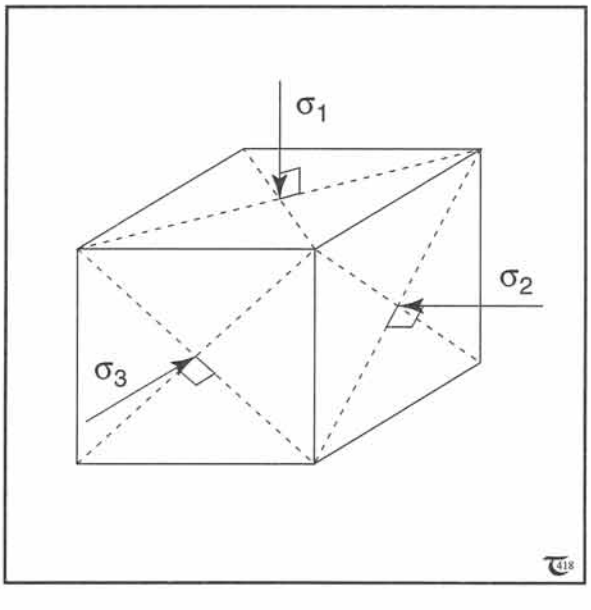

Figure 10-2: Finite rock volume, subjected to uniform stresses, normal to its outer surfaces. Shear stresses cannot occur at these surfaces in this arrangement.

10-1 Ellipsoid and tensor of stress

Stress is a mathematical tensor quantity, and its physical dimension is expressed in Pascals (sections 3-4 & 3-5). The relative magnitude of the elements of stress at any point can be represented *graphically* by an ellipsoid (Fig. 10-1). The three perpendicular axes of the ellipsoid of the total stress are commonly denoted as σ_1, σ_2, and σ_3. The orientation of the principal stresses may be fixed in space by specifying the orientation of the ellipsoid, choosing an appropriate coordinate system. The shape of the stress ellipsoid can be represented *mathematically* by an ellipsoid equation (4-1).

Figure 10-2 shows a finite cube of rock, subjected to a finite state of stress, due to surface forces. If the rock continuum is uniform and without any gradients in mechanical properties, and if no other stress sources exist, the stress inside the volume will be uniform. Each point in the cube is in the same state of stress, and all their stress ellipsoids, thus, are similar. A stress field can be considered as the state of stress in a continuum made up of an infinite number of points, each of which is subjected to a particular stress. Mathematically, the ellipsoid description is practical only if the coordinate axes are chosen parallel to the principal axes of the stress ellipsoid. This is simply so, because, otherwise, equation (4-1) needs to be adapted.

Additionally, if the orientation of the principal stresses is spatially varying (i.e., stress trajectories are curved, see Figure 6-17b), then the principal stress in each location can be conveniently described by an ellipsoid only if the coordinate axes rotate with the trajectory pattern so that the principal stresses remain everywhere parallel to the coordinate axes. This is sometimes impractical; a spatially fixed coordinate system is more useful in practical situations, requiring analysis at a level more advanced than possible with the ellipsoid description. The adoption of a fixed coordinate system means that principal stresses must be decomposed in their tensor components, using the 3x3 matrix notation typical for *second-order tensors* (section 10-4). Recall that scalar units are *zero order-tensor quantities*, represented by a 1x1 matrix, i.e., a single number (cf. section 2-6). Vector quantities are *first-order tensors*, which can be represented in any arbitrary coordinate system as a 1x3 matrix.

□Exercise 10-1: a) Draw stress trajectories for a uniformly-oriented stress field. b) Draw stress trajectories for a heterogeneous stress field. c) Can stress gradients exist in uniformly-oriented stress fields?

10-2 Direction cosines and vectorial index notation

In order later to explain the components of the stress on an arbitrary plane, it is necessary first to introduce here the concepts of direction cosines and vectorial index notation. Any velocity vector, **v**, can be represented in an arbitrary Cartesian coordinate system by its three vector components, v_x, v_y, and v_z. The indices indicate to which coordinate axis the respective component is paralleling (Fig. 10-3a). If the vector, **v**, makes angles α, β, and γ with the coordinate axes X, Y, and Z, respectively, then the magnitude of the vector components can be expressed, using the so-called *direction cosines* ($\cos\alpha$, $\cos\beta$, $\cos\gamma$):

$$v_x = \mathbf{v} \cos \alpha \quad (10\text{-}1a)$$
$$v_y = \mathbf{v} \cos \beta \quad (10\text{-}1b)$$
$$v_z = \mathbf{v} \cos \gamma \quad (10\text{-}1c)$$

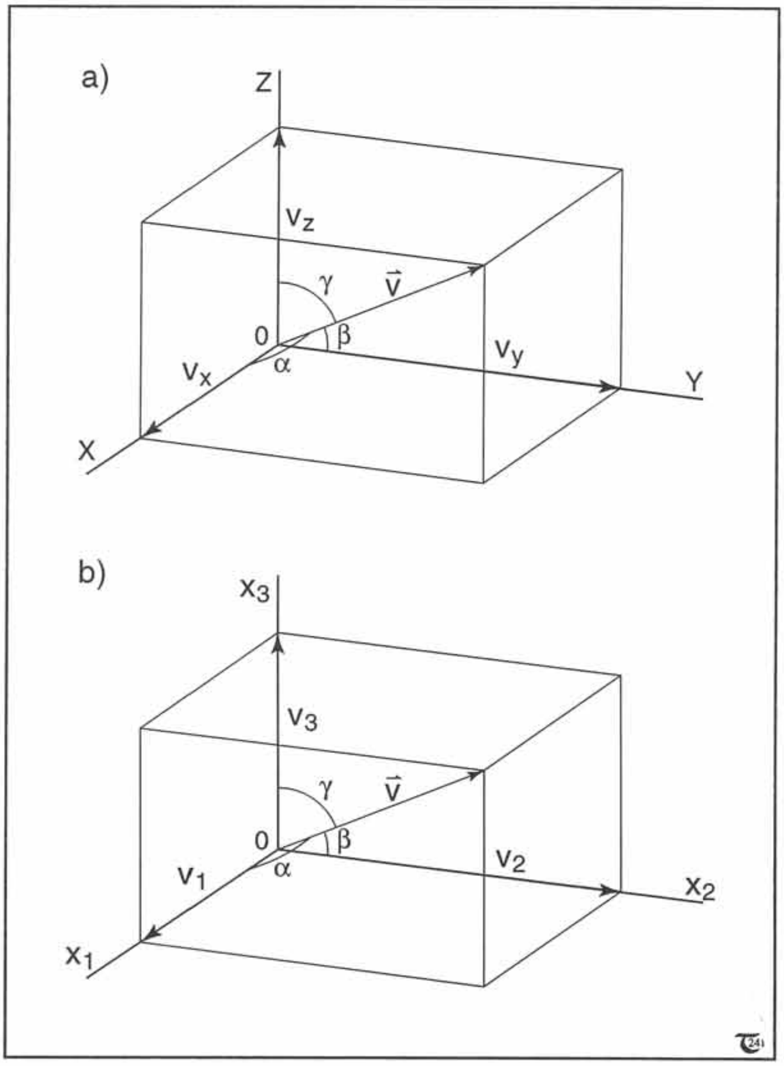

Figure 10-3: *a) & b) Components of velocity vector in XYZ- and $X_1X_2X_3$-spaces.*

The direction cosines are often simply indicated by l, m, and n, with the understanding that $l=\cos\alpha$, $m=\cos\beta$, and $n=\cos\gamma$.

The *absolute magnitude* or length of the total velocity vector, **v**, is related to the vector components by:

$$|\mathbf{v}| = (v_x^2 + v_y^2 + v_z^2)^{1/2} \quad (10\text{-}2)$$

It is common practice in applied mathematics to employ indices to designate vector components.

These indices may be sorted, either *alphabetically* or *numerically*. For example, the vector, **v**, may be represented by its components, v_i, with subscript index i equal to either x, y, and z, or simply 1, 2, and 3. The understanding is that v_x is parallel to the X-axis, v_y is parallel to the Y-axis, and v_z is parallel to the Z-axis (Fig. 10-3a). Likewise, v_1 parallels the X_1-axis, v_2 parallels the X_2-axis, and v_3 parallels the X_3-axis (Fig. 10-3b). The choice of alphabetical or numerical indices is entirely arbitrary, and they may even be used in mixed fashion. The purpose of the index notation is that it significantly shortens the set of equations to describe vector and tensor quantities. For example, the equations (10-1a to 10-1c) may in index notation simply be written as $v_i = l_i \mathbf{v}$.

In what follows, it is useful to recall some practical relationships for the direction cosines. For example, it follows from the Pythagoras theorem that:

$$l^2 + m^2 + n^2 = 1 \quad (10\text{-}3)$$

Another property of direction cosines is that the angle between two lines, α, can be expressed,

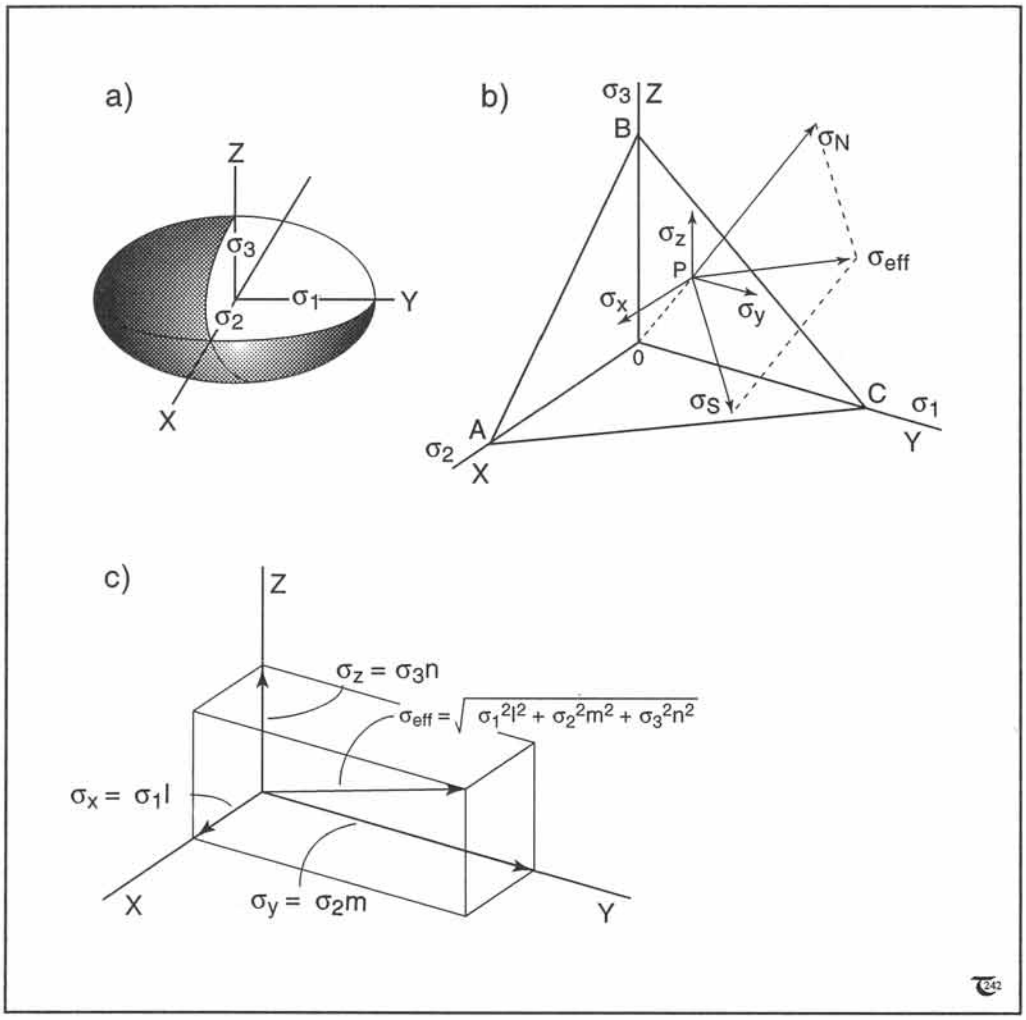

***Figure 10-4:** a) Stress ellipsoid, aligned with coordinate axes, represents state of stress in a point. b) Any plane, ABC, oblique to principal stresses, experiences a normal and shear stress (σ_N and σ_S). c) Definition sketch of effective stress components.*

using their respective orientations (l_1,m_1,n_1) and (l_2, m_2,n_2), as follows:

$$\cos \alpha = l_1l_2+m_1m_2+n_1n_2 \qquad (10\text{-}4a)$$

Consequently, if two lines are perpendicular:

$$l_1l_2+m_1m_2+n_1n_2=0 \qquad (10\text{-}4b)$$

□Exercise 10-2: Write out the set of expressions implied by $v_i=n_iv$.

10-3 Normal and shear stress on an arbitrary plane

Before moving into tensor notation, it is useful to consider stress in a point with coordinate axes, coinciding with the three principal axes of stress (Fig. 10-4a). Any arbitrary plane, ABC, of unit area oblique to the principal stress axes, is subjected to a component of shear and normal stress (Fig. 10-4b). The magnitudes of these two stress components can be obtained as follows: The normal to plane ABC has orientation (l,m,n) or simply n_i, if subscript notation is used for the direction cosines. The magnitude of the principal stress components, adapted for the surface area on which the principal stresses are working, is given by the *effective stress components*, σ_x,σ_y, and σ_z (Fig. 10-4c):

$$\sigma_x=\sigma_1 l \qquad (10\text{-}5a)$$
$$\sigma_y=\sigma_2 m \qquad (10\text{-}5b)$$
$$\sigma_z=\sigma_3 n \qquad (10\text{-}5c)$$

The *effective stress*, σ_{eff}, on the plane, ABC, is:

$$\sigma_{eff}=(\sigma_x^2+\sigma_y^2+\sigma_z^2)^{1/2}=(\sigma_1^2l^2+\sigma_2^2m^2+\sigma_3^2n^2)^{1/2} \qquad (10\text{-}6a)$$

The *normal stress* on the plane, ABC, σ_N, is:

$$\sigma_N=\sigma_1 l^2+\sigma_2 m^2+\sigma_3 n^2 \qquad (10\text{-}6b)$$

The *shear stress* on the plane, ABC, σ_S, can be expressed in terms of the Pythagoras theorem:

$$\sigma_S^2=\sigma_{eff}^2-\sigma_N^2=(\sigma_1-\sigma_2)^2l^2m^2+(\sigma_2-\sigma_3)^2m^2n^2+(\sigma_3-\sigma_1)^2l^2n^2 \qquad (10\text{-}6c)$$

The *planes of maximum shearing stress* can be determined from equation (10-6c), and appear to

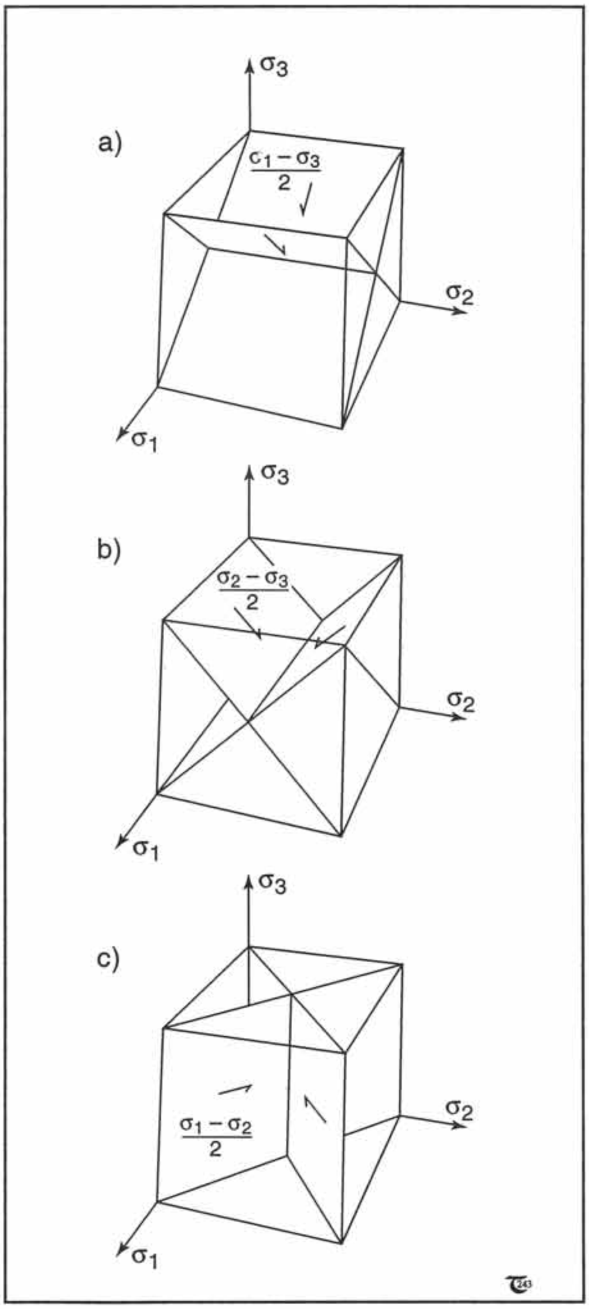

Figure 10-5: *a) to c) Three sets of conjugate planes of maximum shearing stress inside a rock volume subjected to the uniform stress field oriented as shown.*

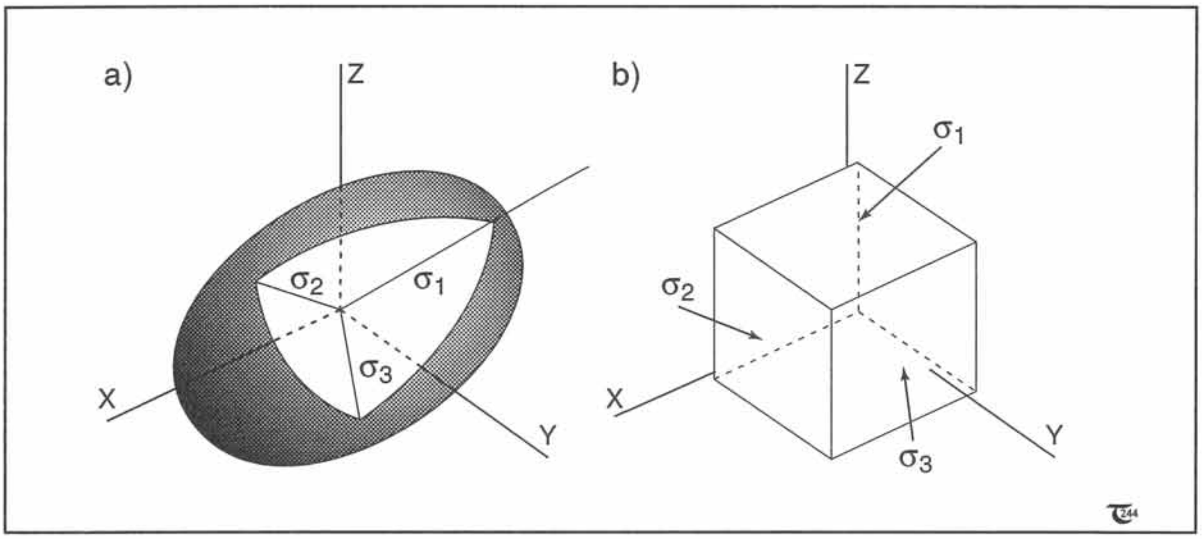

***Figure 10-6:** a) & b) Principal stresses. (a) Stress ellipsoid oriented oblique to the coordinate system. (b) Principal stresses on the principal surfaces of an infinitesimally small unit cube.*

be oriented such that they contain one of the principal stress axes and are at 45° with respect to the other two axes (Fig. 10-5a to c). The magnitudes of the maximum shear stresses for each of these conjugate planes are as follows: $\sigma_S=(\sigma_2-\sigma_3)/2$ (Fig. 10-5a), $\sigma_S=(\sigma_1-\sigma_3)/2$ (Fig. 10-5b), and $\sigma_S=(\sigma_1-\sigma_2)/2$ (Fig. 10-5c).

□Exercise 10-3: Consider a plane with its direction vector or normal oriented at 60° to σ_1 and σ_3 and at 45° to σ_2. The principal stresses are: -600, -400, and -200 MPa. Calculate (a) the total effective stress, (b) the normal stress, and (c) the shear stress on the plane.

10-4 Stress tensor components

The state of total stress in a point is represented by the three principal stress ellipsoid directions, i.e., σ_1, σ_2, and σ_3 (Fig. 10-6a). This state of stress can be expressed in a 3x3 tensor matrix as follows: The stress tensor conventionally represents the stress components exerted by the principal stresses on the faces of an infinitesimally small cubic element. The faces of the cube are always parallel to the principal planes of the coordinate axes used. The Cartesian coordinate system may be oriented arbitrarily with respect to the principal stress axes, as portrayed in Figure 10-6b.

The principal stress, which intersects the cube face normal to the X-axis, is decomposed into one *normal component of stress*, σ_{xx}, and two resolved *shear components*, σ_{xy} and σ_{xz} (Fig. 10-7a). The indices are used such that the first index indicates that the particular component considered is acting on the coordinate plane normal to the X-axis. The second index refers to the orientation in which the component is working: σ_{xx} parallels the X-axis, σ_{xy} parallels the Y-axis, and σ_{xz} parallels the Z-axis (Fig. 10-7a). The alternative, numerical indices are σ_{11}, σ_{12}, and σ_{13} (Fig. 10-7b).

The total stress tensor is as follows:

Oriented parallel to:	X-axis	Y-axis	Z-axis	
Acting on plane normal to X-axis:	σ_{xx}	σ_{xy}	σ_{xz}	
Acting on plane normal to Y-axis:	σ_{yx}	σ_{yy}	σ_{yz}	(10-7)
Acting on plane normal to Z-axis:	σ_{zx}	σ_{zy}	σ_{zz}	

The tensor elements are the vector representations of the principal stresses, expressed in a

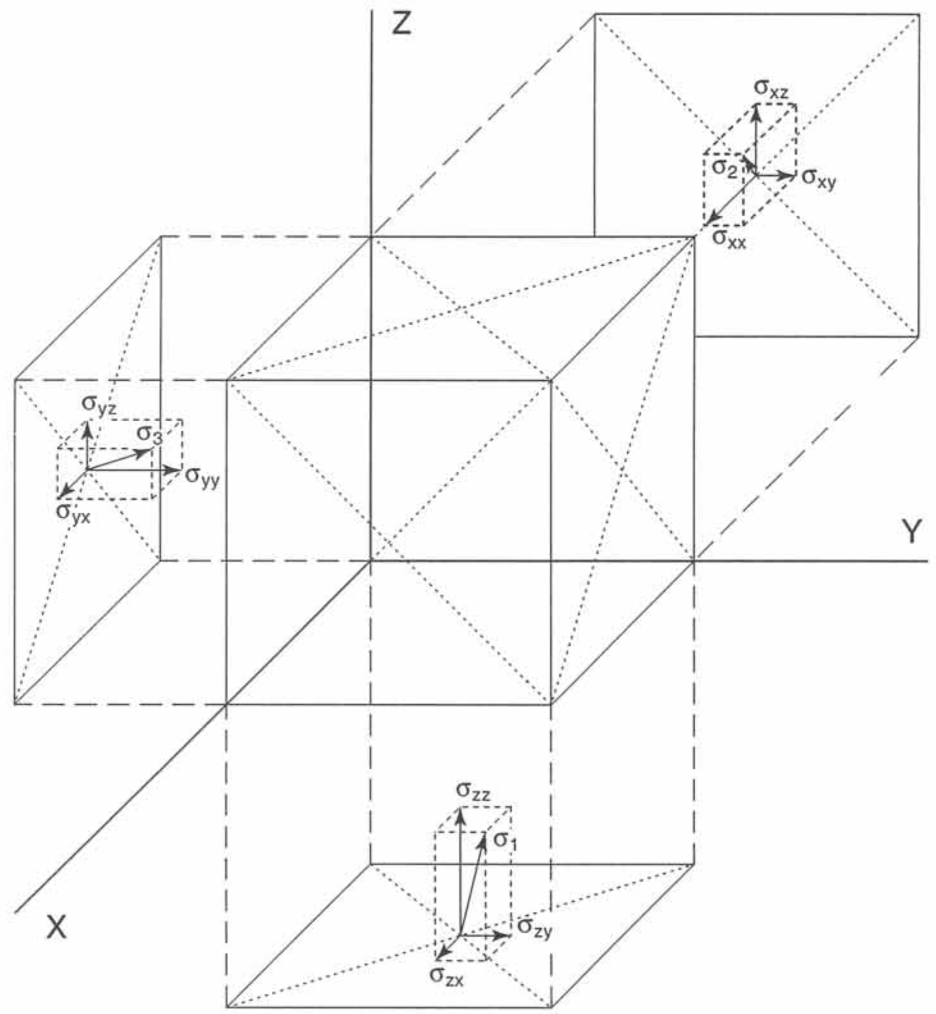

***Figure 10-7a:** Projection of the principal stresses (σ_1, σ_2, σ_3) into nine components, corresponding to the elements of the stress tensor, using alphabetical indices in XYZ-space.*

□Exercise 10-4: The stress tensor is always symmetric. This implies that three of the shear stresses on one side of the diagonal in the stress tensor are identical to the remaining three. a) Specify exactly which shear stresses are the same, using numerical tensor notation. b) Use Figures 10-7a & b to explain why the symmetry of shear stresses exists, assuming that all forces on the cube are balanced. Remember that the cubic volume shown is infinitesimally small.

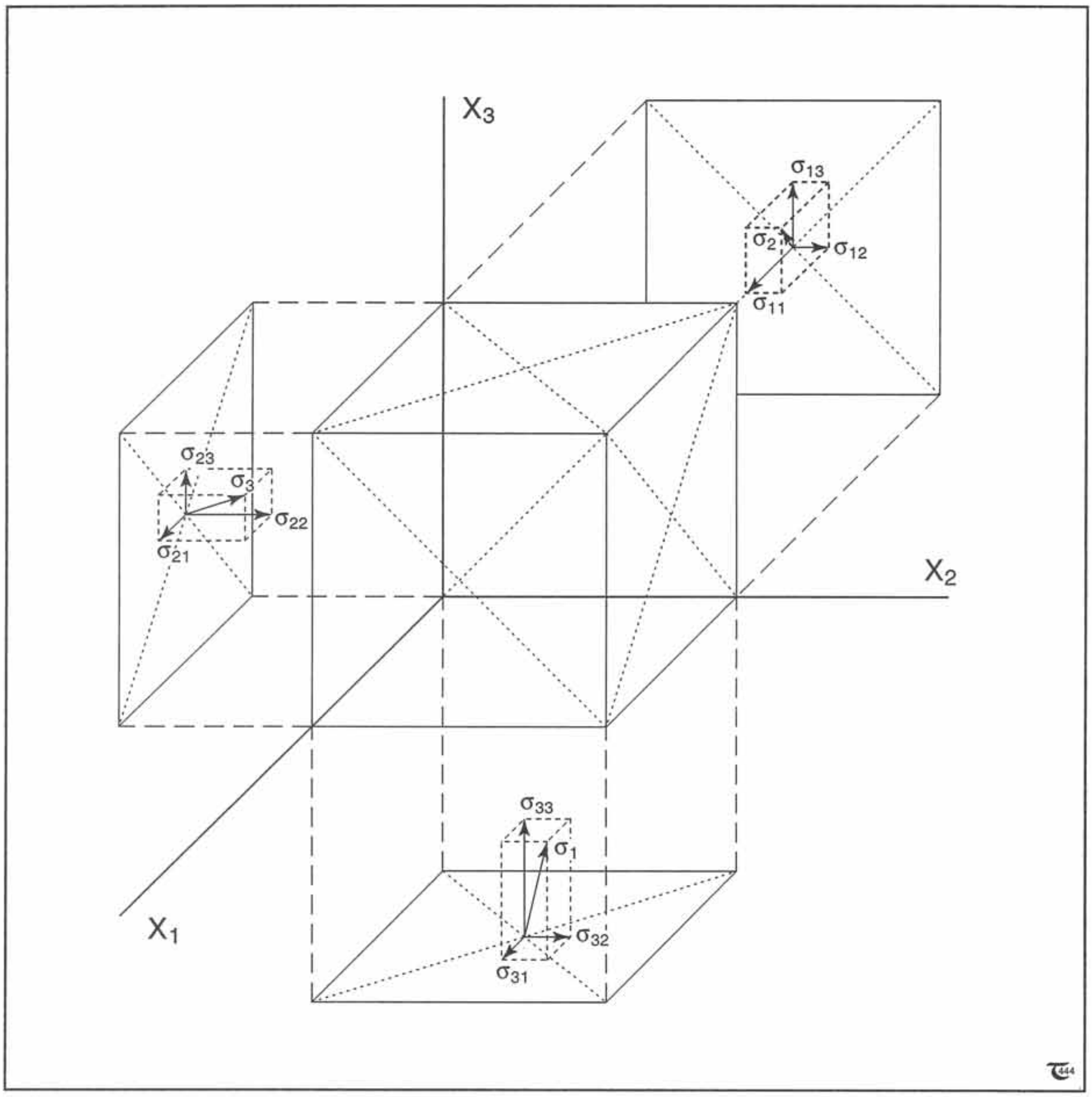

Figure 10-7b: Projection of principal stresses (σ_1, σ_2, and σ_3) into components, described by elements of the stress tensor, using numerical indices of $X_1X_2X_3$-space.

particular coordinate system. They act at a point on three mutually perpendicular planes. All components remain parallel to the coordinate axes. If a different coordinate system is used, then the nine tensor elements will have numerical values different from those for any other coordinate orientation.

10-5 Choice of coordinate axes

If the state of stress in only one point needs to be represented, it is advisable to choose the coordinate axes such that these coincide with the principal stress directions. If the orientation of the coordinate axes is further selected such that principal stresses, σ_1, σ_2, and σ_3, are parallel to the X-, Y-, and Z-axes in this order, then the simplest possible form of the stress tensor is as follows (Fig. 10-8):

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{10-8}$$

with the only non-zero elements $\sigma_{xx}=\sigma_1$, $\sigma_{yy}=\sigma_2$, and $\sigma_{zz}=\sigma_3$.

For a stress state with the principal stress axes in an arbitrary orientation with respect to the coordinate axes, generally, there are nine non-zero elements in the stress tensor. However, of these nine elements, only six are independent because of certain equalities among the shear components, which follows from the consideration that the stress in a point can be maintained only if there is a balance of forces. It follows that $\sigma_{xy}=\sigma_{yx}$, $\sigma_{xz}=\sigma_{zx}$, and $\sigma_{yz}=\sigma_{zy}$. This must be so, because no balance of forces would result if the shear stresses were not *symmetric* in the stress tensor.

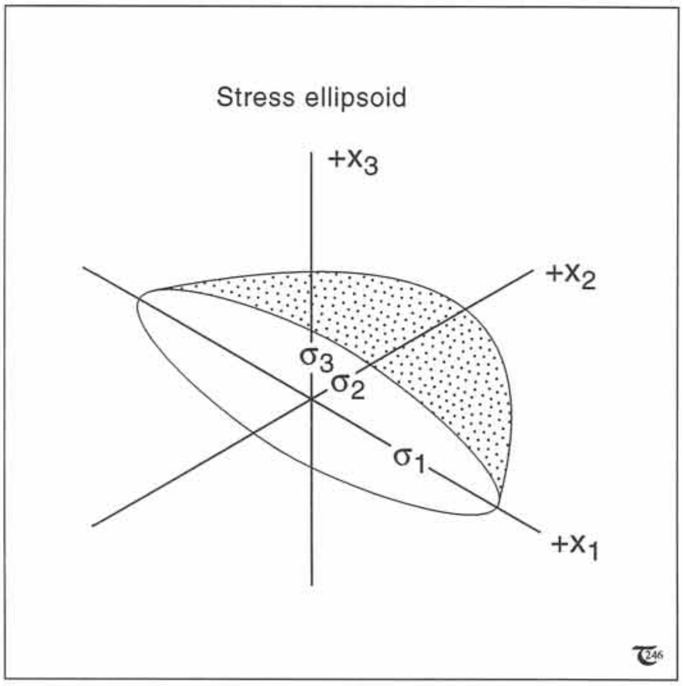

Figure 10-8: *Coincidence of coordinate axes and principal stresses results in stress tensor of the simplest form.*

Table 10-1: *Examples of various notations, used for the elements of the stress tensor.*

Author	***Stress tensor elements***					
Cauchy (early)	A	B	C	D	E	F
Kelvin	P	Q	R	S	T	V
Pearson	xx	yy	zz	yz	zx	xy
Cauchy (late), Maxwell	p_{xx}	p_{yy}	p_{zz}	p_{yx}	p_{zx}	p_{xy}
Neumann, Kirchhoff, Love	X_x	Y_y	Z_z	Y_z	Z_x	X_y
Karman, Timoshenko, Ramsay	σ_x	σ_y	σ_z	τ_{yx}	τ_{zx}	τ_{xy}
Truesdell, Eringen, Clebsch	t_{11}	t_{22}	t_{33}	t_{23}	t_{31}	t_{12}
Green, Zerna	τ_{11}	τ_{22}	τ_{33}	τ_{23}	τ_{31}	τ_{12}
Modern writers	σ_{11}	σ_{22}	σ_{33}	σ_{23}	σ_{31}	σ_{12}
Modern writers	σ_{xx}	σ_{yy}	σ_{zz}	σ_{yx}	σ_{zx}	σ_{xy}
This text:						
Total stress (XYZ-space)	σ_{xx}	σ_{yy}	σ_{zz}	σ_{yx}	σ_{zx}	σ_{xy}
Total stress ($X_1X_2X_3$-space)	σ_{11}	σ_{22}	σ_{33}	σ_{23}	σ_{31}	σ_{12}
Deviatoric stress (XYZ-space)	τ_{xx}	τ_{yy}	τ_{zz}	τ_{yx}	τ_{zx}	τ_{xy}
Deviatoric stress ($X_1X_2X_3$-sp.)	τ_{11}	τ_{22}	τ_{33}	τ_{23}	τ_{31}	τ_{12}

□Exercise 10-5: Determine expressions for all the stress tensor elements for the following situation. σ_2 is parallel to the Y-axis, and σ_1 is at 45° to the X-axis.

10-6 Index notation for tensors

The stress tensor, in *tensor notation* or *index notation*, can simply be represented by σ_{ij}, with normal stress components, σ_{kk}, and shear components, σ_{ij}, for $i \neq j$. The symmetry in the stress tensor implies that elements $\sigma_{ij}=\sigma_{ji}$. A variety of notations has been used to indicate the difference in shear and normal stress (Table 10-1). In this book, τ is reserved for *deviatoric stress* and σ for *total stress*. The subscripts are sufficient to distinguish the normal and shear components. The tensors for total and deviatoric stress are related by:

$$\sigma_{ij}=\tau_{ij}+P_{ij} \qquad (10\text{-}9)$$

with hydrostatic pressure tensor $P_{ij}=P\delta_{ij}=(1/3)\sigma_{kk}\delta_{ij}$. The unit matrix is represented by *Kronecker delta*, δ_{ij}, which is defined as $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$. Generally, in continuum mechanics, it is to be assumed that any given index has a range of three, unless otherwise stated. If a point is subjected to two stress regimes, each represented by a stress tensor, then the state of total stress is equal to the matrix sum of the two stress tensors.

□Exercise 10-6: a) Write out the matrix of Kronecker delta. b) Write out the tensor P_{ij}. c) Write the elements of σ_{ij} in terms of P_{ij} and τ_{ij}.

10-7 Cauchy formula and summation convention

The Cauchy formula is used to calculate the effective stress on an arbitrary plane directly from the stress tensor. It is, therefore, useful to express the components of the effective stress (σ_x, σ_y, and σ_z; c.f. eqs. 10-5) on an arbitrary plane in terms of the tensor components of stress. Consider an arbitrary plane, ABC, of unit area, oblique to the coordinate axes, for which the stress tensor is known (Fig. 10-9a). The components of effective stress are given by the *Cauchy formula*,

□Exercise 10-7: The direction cosines of a plane, ABC, are (0.7, 0.5, 0.5), with $(\alpha,\beta,\gamma)=(45°, 60°, 60°)$. The tensor for total stress is, in MPa:

$$\begin{bmatrix} -40 & -40 & -35 \\ -40 & 45 & -50 \\ -35 & -50 & -20 \end{bmatrix}$$

a) Determine the deviatoric stress tensor. b) Use Cauchy's formula to calculate the components of the effective stress on the plane, ABC. c) Also, calculate the total effective stress, the normal stress, and the shear stress on the plane.

which reads:

$$\sigma_x=\sigma_{xx}l+\sigma_{xy}m+\sigma_{xz}n \qquad (10\text{-}10a)$$
$$\sigma_y=\sigma_{yx}l+\sigma_{yy}m+\sigma_{yz}n \qquad (10\text{-}10b)$$
$$\sigma_z=\sigma_{zx}l+\sigma_{zy}m+\sigma_{zz}n \qquad (10\text{-}10c)$$

The alternative, numerical notation, for the coordinate system in Figure 10-9b, reads:

$$p_1=\sigma_{11}n_1+\sigma_{12}n_2+\sigma_{13}n_3 \qquad (10\text{-}11a)$$
$$p_2=\sigma_{21}n_1+\sigma_{22}n_2+\sigma_{23}n_3 \qquad (10\text{-}11c)$$
$$p_3=\sigma_{31}n_1+\sigma_{32}n_2+\sigma_{33}n_3 \qquad (10\text{-}11c)$$

The effective stress components are here denoted by p_1, p_2, and p_3, because maintenance of σ_i would confuse the effective stress components with the principal stresses. The three set of equations (10-11a to c) can be condensed into a single expression, using tensor notation:

$$p_i=\sigma_{ij}n_j \qquad (10\text{-}12)$$

The full form can be obtained, using *Einstein's double suffix* or *summation convention*, which implies that a repeated suffix represents a sum of three terms, in which that suffix successively takes the values 1, 2, and 3. This means that $\sigma_{ij}n_j$ reads as $\sigma_{i1}n_1+\sigma_{i2}n_2+\sigma_{i3}n_3$.

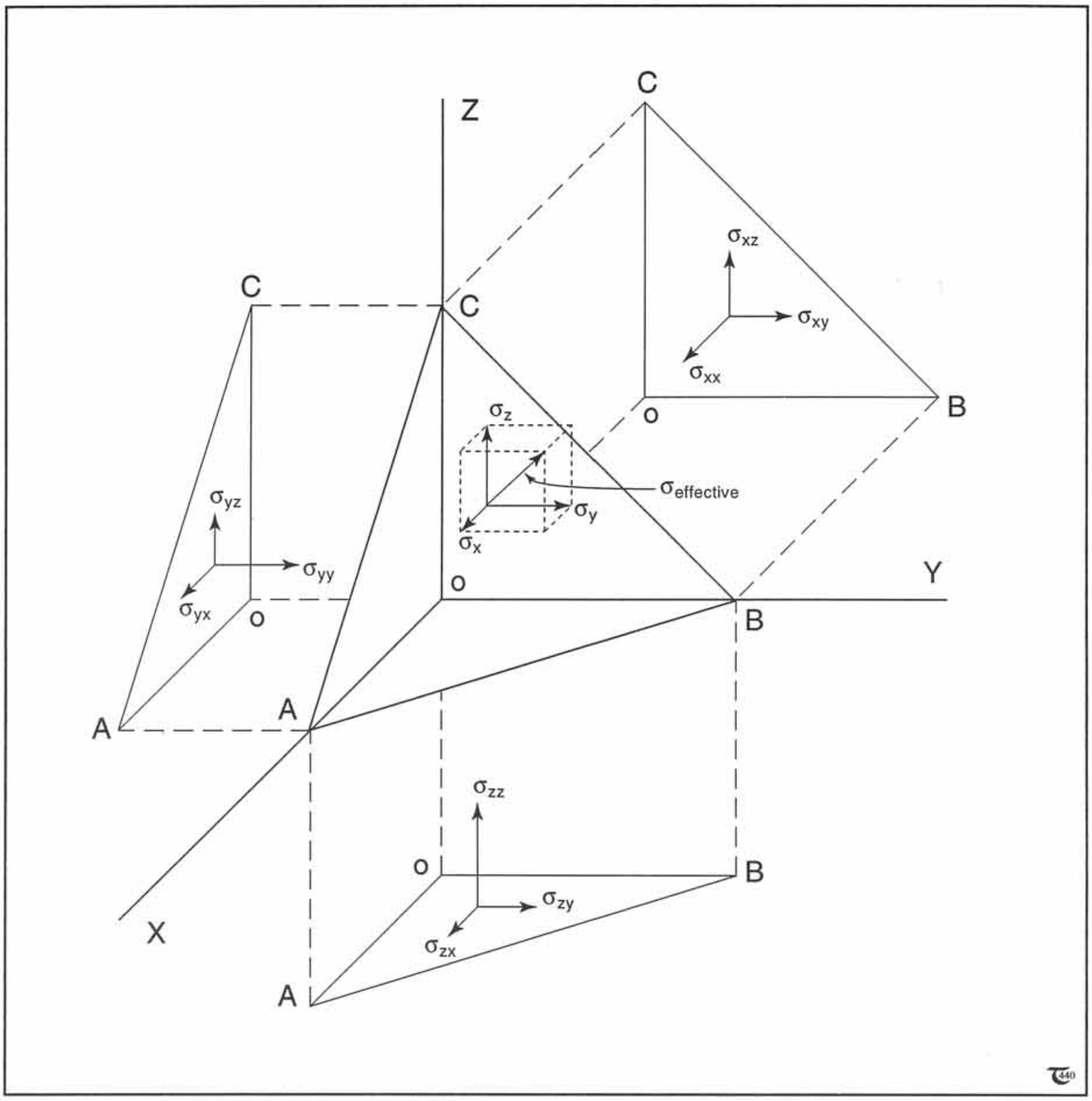

Figure 10-9a: *Components of effective stress (σ_x, σ_y, and σ_z) on an arbitrary plane, ABC, inside an infinitesimally small volume, using alphabetical indices.*

The magnitudes of the two components of shear and normal stress on plane ABC can now be obtained from the effective stress components, as explained in section 10-3 (eqs. 10-6a to c).

10-8 Cubic equation

The cubic equation is used to relate both the orientation and magnitude of principal stresses to their stress tensor notation. Consider the plane, ABC, of Figure 10-9a, but now suppose that the orientation of the plane is varied until the shear-

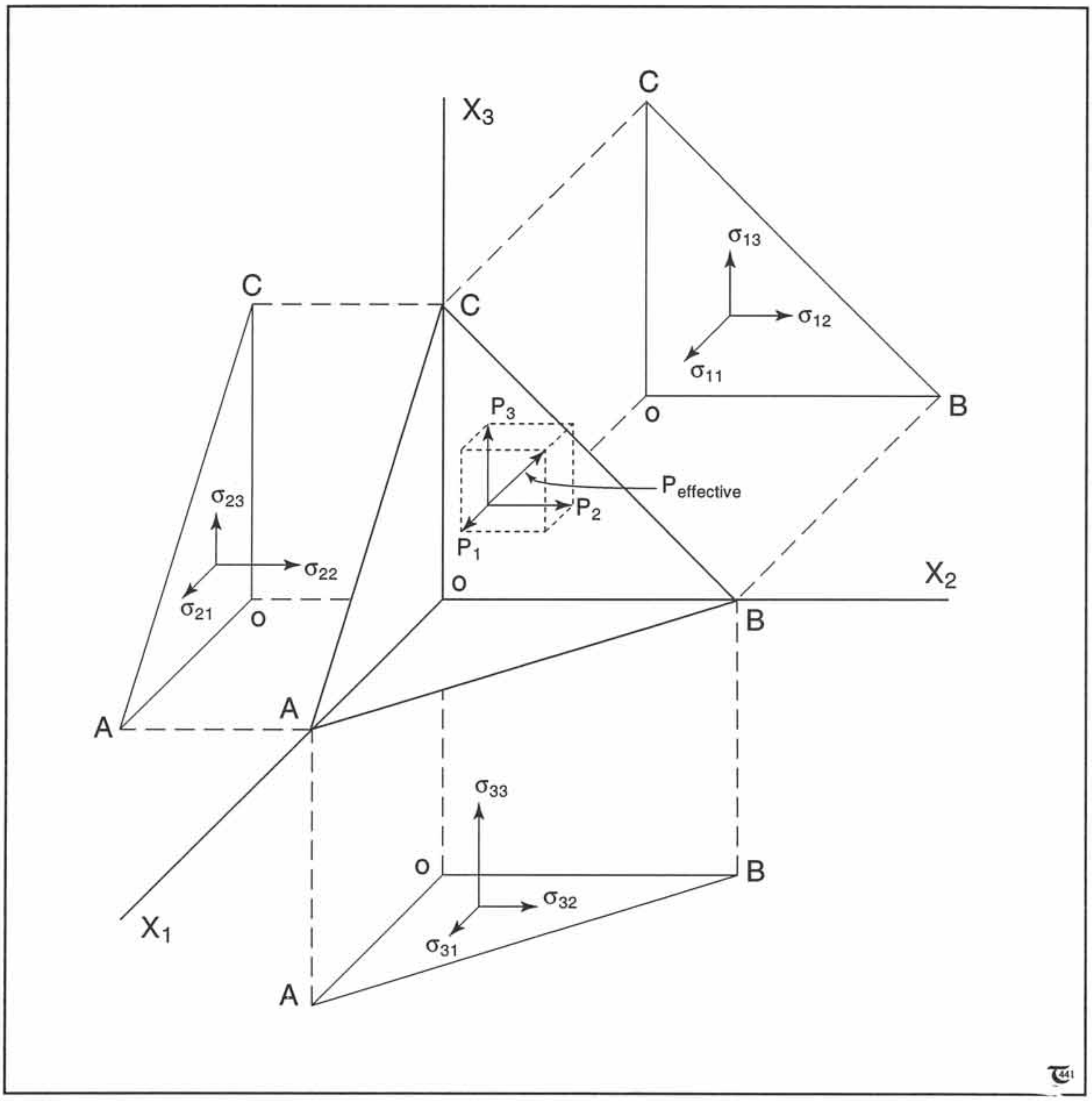

Figure 10-9b: *Components of effective stress (p_1, p_2, and p_3) on an arbitrary plane, ABC, inside an infinitesimally small volume, using numerical indices.*

ing stresses vanish from within the plane. Such a plane will be a principal plane (also termed a *quadratic surface*) and is normal to a principal axis of stress, σ_i. The three effective stress components are:

$$\sigma_x = \sigma_i l \qquad (10\text{-}13a)$$
$$\sigma_y = \sigma_i m \qquad (10\text{-}13b)$$
$$\sigma_z = \sigma_i n \qquad (10\text{-}13c)$$

Figure 10-9b illustrates the same effective stress components in numerical indices. Combining expressions (10-11) and (10-13) yields a cubic equation with three real roots, corresponding to the *magnitude of the three principal stresses*, σ_i, which may be arranged such that $|\sigma_1| \geq |\sigma_2| \geq |\sigma_3|$. The *cubic equation* reads:

$$\sigma_i^3 - \sigma_i^2(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) + \sigma_i(\sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx}-\sigma_{xy}^2-\sigma_{yz}^2-\sigma_{zx}^2) - (\sigma_{xx}\sigma_{yy}\sigma_{zz}+2\sigma_{xy}\sigma_{yz}\sigma_{zx}-\sigma_{xx}\sigma_{yz}^2-\sigma_{yy}\sigma_{zx}^2-\sigma_{zz}\sigma_{xy}^2)=0 \quad (10\text{-}14)$$

The *orientations* (l,m,n) of each principal stress, σ_i, within the currently used coordinate system can be obtained from the stress tensor. The *principal stress orientations* may be obtained, solving the following two equations, complemented with condition (10-3):

$$(\sigma_{xx}\sigma_{zz}-\sigma_{yy}\sigma_i-\sigma_{zz}\sigma_i+\sigma_i^2-\sigma_{yz}^2)/l = (\sigma_i\sigma_{xy}+\sigma_{zx}\sigma_{yz}-\sigma_{xy}\sigma_{zz})/m \quad (10\text{-}15a)$$

$$(\sigma_i\sigma_{xy}+\sigma_{zx}\sigma_{yz}-\sigma_{xy}\sigma_{zz})/m = (\sigma_i\sigma_{zx}+\sigma_{xy}\sigma_{yz}-\sigma_{zx}\sigma_{yy})/n \quad (10\text{-}15b)$$

Both the magnitude and orientation of the principal stresses can, thus, be obtained from the stress tensor, using equations (10-3 and 10-13 to 10-15).

□Exercise 10-8: Calculate: (a) the magnitude and (b) the orientation of the principal stresses at a point with a stress tensor as given in exercise 9-8.

10-9 Tensor transformation formula

It is outlined here how and why the tensor transformation formula is used to obtain the tensor of a particular state of stress within different coordinate systems. The numerical value of the elements of each stress tensor are valid only for a particular coordinate system. It is, therefore, useful to be able to transform stress tensors from one coordinate system to another. This is possible, using a relatively simple *tensor transformation formula*. The two sets of coordinate axes are illustrated in Figure 10-10, and Table 10-2 specifies the associated set of direction cosines, n_{ij}, required to describe the angular relationships between the new and the old coordinate system. Let σ_{ij}^* and σ_{pq} be the stress tensor elements in the new and old coordinate systems, respectively.

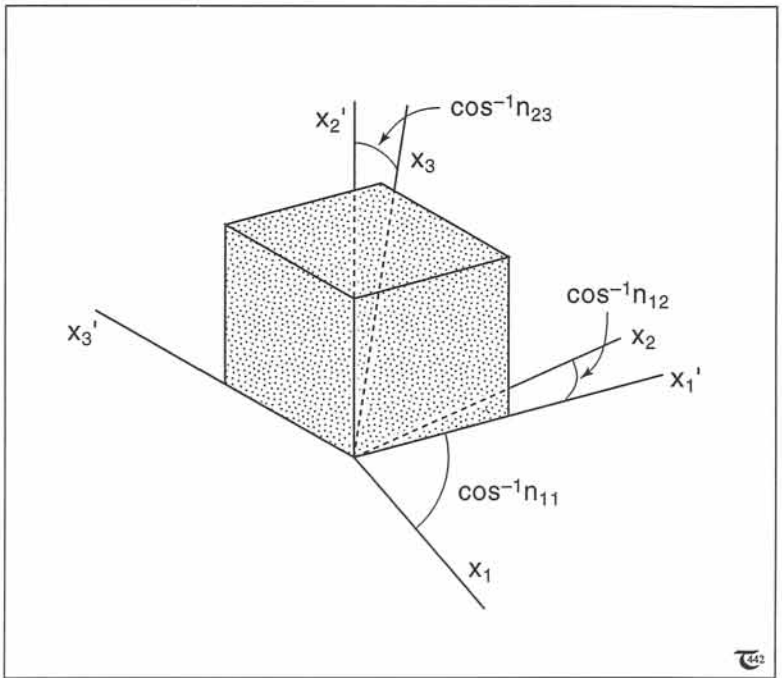

Figure 10-10: *Three of the nine direction cosines between new coordinate axes X_i' and old coordinate system X_i are indicated. Table 10-2 contains all nine direction cosines needed for the coordinate transformation.*

Table 10-2: *Direction cosines for transformation of coordinate system.*

	Old axes		
New axes	X_1	X_2	X_3
X_1'	n_{11}	n_{12}	n_{13}
X_2'	n_{21}	n_{22}	n_{23}
X_3'	n_{31}	n_{32}	n_{33}

They are then related by the following tensor transformation formula:

$$\sigma_{ij}^* = n_{ip}\, n_{jq}\, \sigma_{pq} \quad (10\text{-}16)$$

The conventional formula for σ_{11}^* would read:

$$\sigma_{11}^* = n_{11}n_{11}\sigma_{11}+n_{11}n_{12}\sigma_{12}+n_{11}n_{13}\sigma_{13} + n_{12}n_{11}\sigma_{21}+n_{12}n_{12}\sigma_{22}+n_{12}n_{13}\sigma_{23} + n_{13}n_{11}\sigma_{31}+n_{13}n_{12}\sigma_{32}+n_{13}n_{13}\sigma_{33} \quad (10\text{-}17)$$

Similarly, the other stress elements can be developed from equation (10-16), using the tensor summation convention.

□Exercise 10-9: A coordinate system is changed such that the new axes coincide with the old axes, but is rotated such that $X_1^*=X_2$, $X_2^*=X_3$, and $X_3^*= X_1$. If the initial stress tensor elements are σ_{pq}, write the new tensor, σ_{ij}^*, in terms of the old tensor elements.

□Exercise 10-10: The deviatoric stress tensor for an area of uniform stress is $\tau_{xx}=100$ MPa, $\tau_{zz}=-100$ MPa, and all other elements are zero. a) Calculate the stress tensor for this area, rotating the coordinate axes 45° anti-clockwise about X_2. b) What happened to the normal stresses? c) What is the magnitude of the maximum shear stress?

10-10 Plane stress in two dimensions

Consider a case where the state of stress is two-dimensional in the sense that there are no surface forces in the Y-direction. In practice this situation is possible only for deviatoric stresses, and it arises when $P=\sigma_2$, so that $\tau_2=0$. The only non-zero elements of the deviatoric stress tensor, choosing an appropriate coordinate system (Fig. 10-11a), are in the XZ-plane. The *deviatoric stress tensor* will read:

$$\begin{bmatrix} \tau_{xx} & 0 & \tau_{xz} \\ 0 & 0 & 0 \\ \tau_{zx} & 0 & \tau_{zz} \end{bmatrix} \qquad (10\text{-}18)$$

with the balance of forces implying that $\tau_{xz}=\tau_{zx}$. The state of total stress in a plane with normal inclined at angle θ with respect to the X-axis is as follows: The total *normal and shear stresses* on plane AB can be found through the coordinate transformation, illustrated in Figure 10-11b. The normal and shear stresses are:

$$\sigma_{xx}^*=\sigma_{xx}\cos^2\theta+\sigma_{zz}\sin^2\theta+\sigma_{xz}\sin 2\theta \qquad (10\text{-}19a)$$
$$\sigma_{xz}^*=(1/2)(\sigma_{zz}-\sigma_{xx})\sin 2\theta+\sigma_{xz}\cos 2\theta \qquad (10\text{-}19b)$$

The *orientation of the principal stresses* is found by setting σ_{xz}^* in equation (10-19b) to zero. The direction of θ will then define a principal plane perpendicular to a principal axis of stress:

$$\tan 2\theta=2\sigma_{xz}/(\sigma_{xx}-\sigma_{zz}) \qquad (10\text{-}20)$$

Substitution of expression (10-20) into (10-19a) and some goniometric manipulation yields an expression for finding the *magnitude of the principal stresses* from the tensor elements:

$$\sigma_1=[(\sigma_{xx}+\sigma_{zz})/2]+[[(\sigma_{xx}-\sigma_{zz})^2/4]+\sigma_{xz}^2]^{1/2} \qquad (10\text{-}21a)$$
$$\sigma_3=[(\sigma_{xx}+\sigma_{zz})/2]-[[(\sigma_{xx}-\sigma_{zz})^2/4]+\sigma_{xz}^2]^{1/2} \qquad (10\text{-}21b)$$

The Mohr expressions for *normal and shear stress on an arbitrary plane* can now be derived by considering the X- and Z-axes in Figure 10-11a to be principal axes so that $\sigma_1=\sigma_{xx}$, $\sigma_3=\sigma_{zz}$, and $\sigma_{xz}=0$. Substitution in expressions (10-19a & b) yields:

$$\sigma_{xx}^*=[(\sigma_1+\sigma_3)/2]+[(\sigma_1-\sigma_3)/2]\cos 2\theta \qquad (10\text{-}22a)$$
$$\sigma_{xz}^*=[(\sigma_1-\sigma_3)/2]\sin 2\theta \qquad (10\text{-}22b)$$

Differentiating equation (10-19b) with respect to θ and equating the resulting expression to zero yield the angle θ for the *conjugate planes on which the shear stress is at maximum*:

$$\tan 2\theta=(\sigma_{zz}-\sigma_{xx})/2\sigma_{xz} \qquad (10\text{-}23)$$

Comparison of expressions (10-20) and (10-23) shows that the planes of maximum shear stress are at 45° to the principal stress axes. The *maximum shear stress magnitude* in those planes follows from equation (10-22b) by letting θ be 45°:

$$(\sigma_{xz})_{max}=(\sigma_1-\sigma_3)/2 \qquad (10\text{-}24)$$

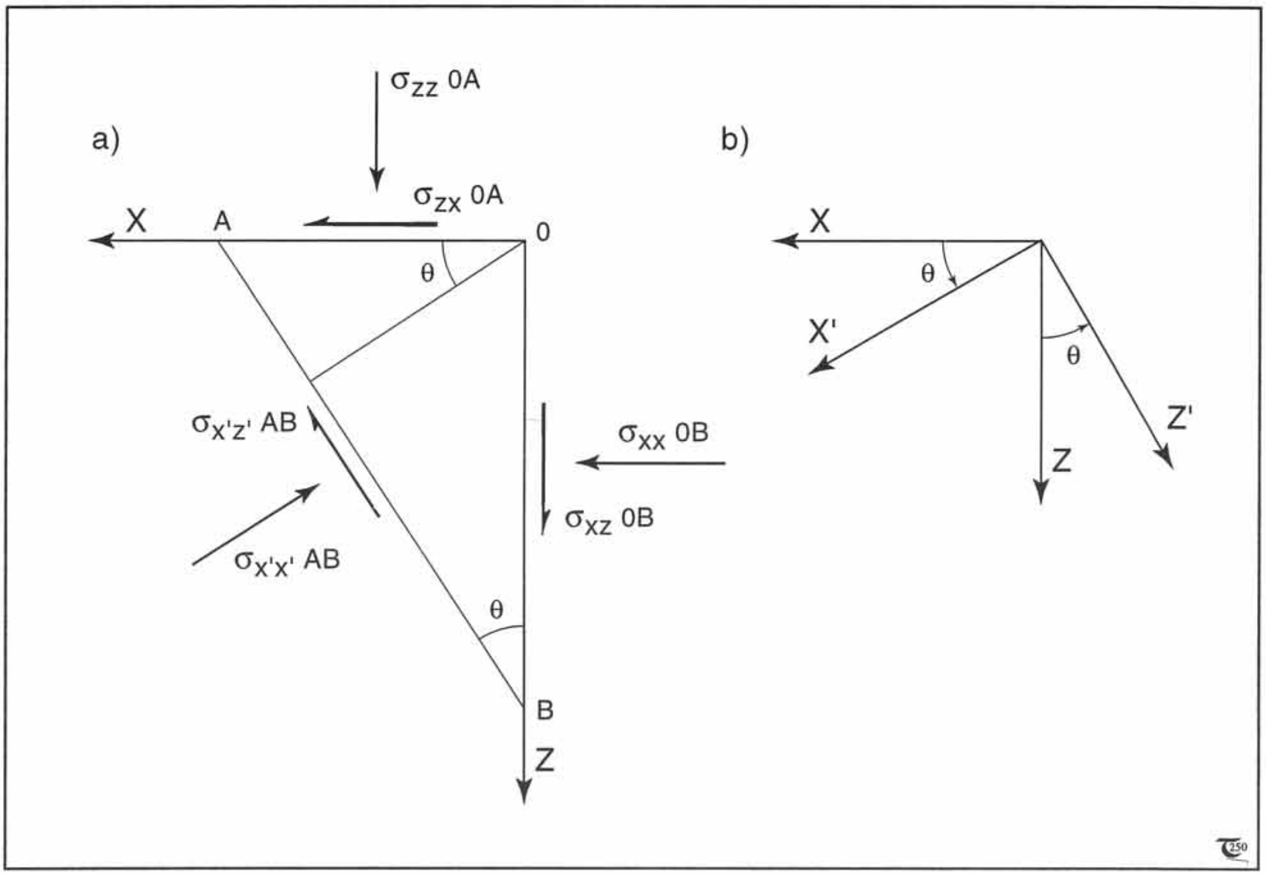

Figure 10-11: a) Normal and shear stresses on inclined plane AB can be found by the coordinate transformation shown in (b). The depth axis, Z, is rotated into parallelism with the dip of plane AB.

The maximum shear stress is half the difference of the principal stresses.

□**Exercise 10-11: The state of stress at a point on a fault plane is σ_{zz}=-150 MPa, σ_{xx}=-200 MPa, and σ_{xz}=0. The depth scale is along Z, and X is pointing westward. What are: (a) the normal stress and (b) the shear stress on the fault plane if the fault strikes N-S and dips 35° to the west?**

□**Exercise 10-12: Given the principal stresses -45 MPa and -80 MPa with σ_1 horizontal and σ_3 vertical. What is the state of stress on a bedding surface, dipping 30°?**

□**Exercise 10-13: An overcoring stress measurement in a mine at a depth of 1.5 km gives normal stresses of -62 MPa in the N-S direction, -48 MPa in the E-W direction, and -51 MPa in the NE-SW direction. Determine the magnitudes and directions of the principal stresses.**

□Exercise 10-14: The measured horizontal principal stresses at a depth of 200 m are given below as a function of distance to the San Andreas Fault. What are the values of the maximum shear stress at each distance?

Distance (km)	σ_{max}(MPa)	σ_{min} (MPa)
2	9	8
4	14	8
22	18	8
34	22	11

10-11 Stress tensor invariants

It follows from the tensor transformation equations that a stress tensor is generally valid only for a particular orientation of a coordinate system. An exception is P_{ij}, which is identical in all coordinate systems, but this is the matrix representation for the scalar quantity of pressure, P, and, therefore, is not a typical 2nd order tensor. However, even true stress tensors have some properties that are independent of the coordinate frame used to describe them. These so-called stress-invariants are relationships obtained from the cubic equation (10-14), which may be solved to find the three real roots, σ_1, σ_2, and σ_3, known as the principal stresses. The cubic equation, therefore, holds for any coordinate system. This means that for one particular orientation (i.e., when the coordinate axes coincide with the principal axes of stress) all shear stresses vanish and the normal stresses are equal to σ_1, σ_2, and σ_3. Substitution of these conditions into the cubic equation (10-14) yields:

$$\sigma_i^3-\sigma_i^2(\sigma_1+\sigma_2+\sigma_3)+ \sigma_i(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)-\sigma_1\sigma_2\sigma_3=0 \qquad (10\text{-}25a)$$

or simply:

$$(\sigma_i-\sigma_1)(\sigma_i-\sigma_2)(\sigma_i-\sigma_3)=0 \qquad (10\text{-}25b)$$

The three invariants, I_1, I_2, and I_3, are defined as the coefficients of equation (10-25a):

$$\sigma_i^3-\sigma_i^2 I_1+\sigma_i I_2-I_3=0 \qquad (10\text{-}26)$$

The individual invariants in terms of principal stresses read:

$$I_1=\sigma_1+\sigma_2+\sigma_3 \qquad (10\text{-}27a)$$
$$I_2=\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1 \qquad (10\text{-}27b)$$
$$I_3=\sigma_1\sigma_2\sigma_3 \qquad (10\text{-}27c)$$

In terms of the stress tensor elements:

$$I_1=\sigma_{xx}+\sigma_{yy}+\sigma_{zz} \qquad (10\text{-}28a)$$
$$I_2=\sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx}-\sigma_{xy}^2-\sigma_{yz}^2-\sigma_{zx}^2 \qquad (10\text{-}28b)$$
$$I_3=\sigma_{xx}\sigma_{yy}\sigma_{zz}+2\sigma_{xy}\sigma_{yz}\sigma_{zx}-\sigma_{xx}\sigma_{yz}^2- \sigma_{yy}\sigma_{zx}^2-\sigma_{zz}\sigma_{xy}^2 \qquad (10\text{-}28c)$$

One important aspect of the stress invariants is their physical meaning. For example, $I_1=0$ if there is no confining pressure so that all stresses are effectively deviatoric. The second stress invariant is a measure for the presence of shear stresses. The third stress invariant vanishes if the stress is two-dimensional. Another important role of the stress invariants is that they provide a method for finding the principal stresses from six known stress tensor elements. The three invariants are calculated first to define the cubic equation, which can then be solved for the principal stresses. The invariants of the stress tensor also are important in the Von Mises failure criterion of three-dimensional yield surfaces in frictional plasticity.

□Exercise 10-15: Calculate the magnitude of the principal stresses for a state of stress with the three invariants as folows: I_1=4 MPa. I_2=-11 MPa, and I_3=-30 MPa.

> □**Exercise 10-16: a) Show that the cubic equation simplifies significantly for two-dimensional or plane stress without a pressure. b) For such cases, give the solutions of the cubic equation in terms of the second invariant of the stress tensor.**

References

A. Books

It is recommended for students of rock mechanics to touch base with classical texts of continuum mechanics. This will improve their grasp of the physics behind the theory. General texts on mechanics, including advanced concepts on tensor computation of stress, are listed below:

Mechanics of Continua (1967, Wiley, New York), by A.C. Eringen.

A First Course in Continuum Mechanics (1969, Prentice-Hall), by Y.C. Fung.

Introduction to the Mechanics of a Continuous Medium (1969, Prentice-Hall), by C. Truesdell.

Stress and Deformation (1995, Oxford University Press), by G. Oertel. A handbook on tensors in Geology.

B. Articles

Some considerations on the relationship between deformation patterns, including both brittle and ductile structures, and the orientation of the principal stress are outlined in the following articles:

Angelier, J. (1979, *Tectonophysics*, volume 56, pages T17 to T26). Determination of the mean principal directions of stress for a given fault population.

Weijermars, R. (1991, *Journal of Structural Geology*, volume 13, pages 1,061 to 1,078). The role of stress in ductile deformation.

Weijermars, R. (1993, *Geological Society of America Bulletin*, volume 105, pages 1,491 to 1,510). Estimation of paleostress orientation within deformation zone between two mobile plates.