III. 3. SPIN HAMILTONIAN IN AN EXTERNAL MAGNETIC FIELD: MEAN-FIELD APPROXIMATION

Consider an Ising Hamiltonian for a spin S_i, $\hat{H} = -JS_i \sum_j S_j - h_0 \vec{S}_i$. Here the spins S_i can only assume values $\pm 1/2$, $J > 0$ is a ferromagnetic coupling constant, the summation is carried out over all sites j that are nearest neighbors to i (the coordination number — the number of nearest neighbors — equals N), and $h_0 > 0$ is the external magnetic field. We measure magnetic field for convenience in energy units ($\mu_B = 1$).

(a) Below critical temperature and in not too high fields h_0 there are three solutions to the mean field equations. The middle one is unstable; two others are stable. The solution with a lower energy will be chosen. Calculate the energy of both solutions and determine their relative stability.

(b) Above the critical temperature, there is only one solution. In low external fields, $h_0 \to 0$, the average spin is proportional to the external field, $\langle S_i \rangle = \chi h_0$. Caluclate the succeptibility χ.

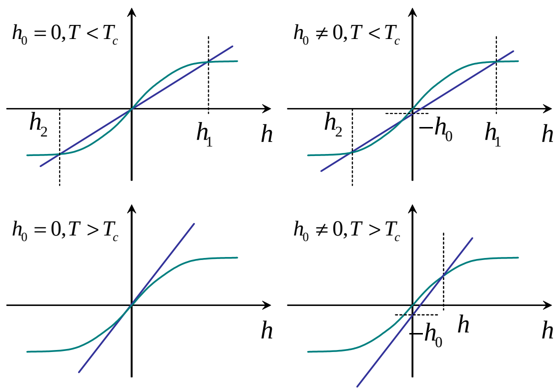

FIG. 3: Graphical solution of Eq.(5).

In the mean-field approximation, one replaces all the spins interacting with a given spin by the averaged field, $\sum_j S_j \to N\langle S_j \rangle$. The energy (Hamiltonian) of the system takes the form

$$\hat{H} = -hS_i, \quad h = h_0 + N\langle S_j \rangle \; . \tag{4}$$

The average spin $\langle S_i \rangle$ is found self-consistently. Indeed, the probability for spin-up depends exponentially on the energy of spin-up configuration, $P_\uparrow \propto \exp(h/2k_BT)$. In the same way, $P_\downarrow \propto \exp(-h/2k_BT)$. But the sum of these two probabilities equals one, and thus

$$P_\uparrow = \exp(h/2k_BT)/2\cosh(h/2k_BT) \; ; \quad P_\downarrow = \exp(-h/2k_BT)/2\cosh(h/2k_BT) \; .$$

The average spin becomes $\langle S_i \rangle = 1/2(P_\uparrow - P_\downarrow) = (1/2)\tanh(h/2k_BT)$. This gives us the equation for the field h,

$$h = h_0 + \frac{JN}{2} \tanh \frac{h}{2k_BT} \; . \tag{5}$$

This equation can not be solved analytically, but we can investigate the main features. First, everything is clear in zero external field, $h_0 = 0$. Then the right hand side of Eq. (5) is linear in low fields, $\tanh(h/2k_BT) \approx h/2k_BT$. When the slope of the right-hand side is greater than one ($T < T_c \equiv JN/(4k_B)$), the equation has three solutions. One is $h = 0$, corresponding to a non-magnetic system ($\langle S_i \rangle = 0$). Two other equations correspond to the non-zero average spins — the system is magnetic. For higher temperatures $T > T_c$, there is only one solution, corresponding to zero magnetization — the system is in a paramagnetic (not ferromagnetic) state.

(a) If the external field h_0 is weak, one still has three solutions for $T < T_c$. The middle one (small h_0) corresponds to the maximum of energy; the other two correspond to two energy minima. For $h_0 = 0$ these two would be completely equivalent: Indeed, there is no preferential direction of magnetization, and if $h \neq 0$ is a solution, $-h$ is also a solution with the same energy: One can flip magnetization at no energy cost. For $h_0 > 0$, the situation is different: there are two solutions $h_1 > 0$ and $h_2 < 0$, with $|h_1| > |h_2|$. Let us compare their energies.

The energy is the averaged Hamiltonian, thus $E = -h\langle S_i \rangle = -(h/2)\tanh(h/2k_BT)$. This is an even function of h, and decreases monotonously for $h > 0$. Then $E(h_1) < E(h_2)$: the lower energy corresponds to the state with magnetization aligned with external field.

(b) Above the critical temperature, there is only one solution close to $h \approx 0$. In Eq. (5), let us expand the right-hand side in h. Then we can solve the equation,

$$h = \frac{h_0}{1 - JN/(4k_BT)} \, .$$

The average spin is $\langle S_i \rangle = (1/2)\tanh(h/2k_BT) \approx h/(4k_BT)$. It is proportional to the external field h_0, $\langle S_i \rangle = \chi h_0$. Writing the succeptibility χ in terms of the ternsition temperature, we find $\chi = 1/(4k_B(T - T_c))$, in accordance with general conclusions for the temperature dependence of the succeptibility that follow from the Landau theory of second order phase transitions.