Advanced Solid State Physics: Problems

I. 4. BOSE-CONDENSATION

Consider ideal Bose gas of spinless particles at the temperature T.

(a) Express the number of particles as the function of temperature and chemical potential.

(b) Keeping the number of particles fixed, express the chemical potential as the function of temperature. At what temperature does it turn to zero? (*Temperature of bose condensation*).

(c) For temperature below Bose condensation, a macroscopic number of particles is in the state with zero energy (*condensate*). From the condition that the chemical potential is zero, find how the temperature dependence of the number of particles in the condensate.

(a) The number of particles in the Bose gas can be expressed in a way similar to the number of particles in the Fermi gas,

$$N = \sum_i f_B(E_i) ,$$

where $f_B(E) = (\exp((E-\mu)/k_BT) - 1)^{-1}$ is Bose-Einstein distribution function. In the continuous limit, we replace $\sum_i \to V \int d^3p/(2\pi\hbar)^3$, where V is the total volume of the system,

$$N = \frac{1}{(2\pi\hbar)^3} V \int d^3p f_B(p) = \frac{(k_BTm)^3/2}{\sqrt{2}\pi^2\hbar^3} V \int_0^\infty \frac{\sqrt{z}dz}{\exp(z - \mu/k_BT) - 1} , \tag{1}$$

where we have replaced the integration over momenta by the integration over energy, and introduced the dimensionless energy $z = E/k_BT$. Eq. (1) gives the dependence of the number of particle on temperature and chemical potential. To say it differently, if the number of particles is fixed, it provides the temperature dependence of the chemical potential. Obviously, very high temperatures correspond to very high negative value of the chemical potential, so that the exponential in the denominator is small. If one lowers the temperature, the chemical potential grows and at certain temperature turns to zero.

(b) Bose condensation temperature T_c can be found from Eq. (1) if we put $\mu = 0$. The integral in the right-hand side is just a number. It can be expressed in terms of the zeta-function; putting all numerical constants together, we obtain

$$T_c = 3.31 \frac{\hbar^2}{m} \left(\frac{N}{V}\right)^{2/3} .$$

Note that it is higher for high densities.

(c) At temperatures below T_c the chemical potential, found from Eq. (1), is negative. This contradicts to the general principle of statistical physics, stating that the chemical potential of a system of Bose particles must always be negative — otherwise the filling factor of the lowest energy state would be negative.

The paradox can be resolved if we realize that at zero temperature bosons would all fall in the lowest energy state ($E = 0$). This is the difference with fermions — for the fermions Pauli principle forbids such behavior, and the ground state is filled Fermi sphere. Thus, a macrosopic number of particles at low temperatures occupy the state with $E = 0$. But the contribution of low energies to the number of particles expressed by Eq. (1) is zero. Thus, we need to count them separately. Note that the chemical potential of our system below Bose condensation temperature has to be zero — the energy does not depend on the number of particles, since addition of any number of particles in the lowest state does not change the energy. We obtain

$$N = N_0 + \frac{(k_BTm)^{3/2}}{\sqrt{2}\pi^2\hbar^3} V \int_0^\infty \frac{\sqrt{z}dz}{\exp z - 1} ,$$

where N_0 is the number of particles in the state $E = 0$ — *Bose condensate*. This gives

$$N_0 = N \left[1 - \frac{T}{T_c}\right]^{3/2} .$$

At zero temperatures, all particles are in the condensate.