IV. 5. UPPER CRITICAL FIELD IN TYPE II SUPERCONDUCTORS

Using Ginzburg-Landau equations for $\kappa > 1/\sqrt{2}$, find at which magnetic field even an infinitesimal piece of a superconductor in the bulk of the sample becomes unstable.

Let us choose the z coordinate axis parallel to the magnetic field, and take the following gauge for the vector potential, $A_y = Hx$, $A_x = A_z = 0$. We describe the superconducting piece far from the boundary (actually, a possibility of creation of a superconducting piece close to the boundary would yield a higher critical field). For simplicity, let us consider a piece with the wave function that only depends on one coordinate, x (it turns out that investigation of a superconducting piece of an arbitrary shape gives the same result for the critical field). Ginzburg-Landau equations in dimensionless variables have the following form,

$$\frac{1}{\kappa^2}\frac{d^2\Psi}{dx^2} + \Psi(1-A^2) - \Psi^3 = 0;$$

$$\left.\frac{d\Psi}{dx}\right|_{boundary} = 0;$$

$$\frac{d^2 A}{dx^2} - \Psi^2 A = 0 \ ,$$

where we assumed that Ψ is real (the phase of Ψ cancels anyway). If the superconducting piece is far from the boundary, the order parameter decays off the piece, and is very small at the boundary, thus the boundary condition is satisfied automatically. Since we are looking for the infinitelesimal piece, the order parameter must be small, $\Psi \ll 1$, and thus the term with Ψ^3 in the first equation can be disregarded, as well as the term with $\Psi^2 A$ in the second equation. We are thus left with the following equations,

$$\frac{1}{\kappa^2}\frac{d^2\Psi}{dx^2} + \Psi(1-A^2) = 0;$$

$$\frac{d^2 A}{dx^2} = 0 \ .$$

The second equation gives that A is a linear function of x, and, since at the transition the field must fully penetrate the sample (Meissner effect disappears), one has $A_y = Hx$. We plug this in the first equation, to obtain

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + \frac{kx^2}{2}\Psi = \epsilon\Psi \ , \tag{6}$$

where we have identified $\epsilon = 1$, $2m/\hbar^2 = \kappa^2$, $k = 2H^2$. However, Eq. (6) is Schrödinger equation for a harmonic oscillator, and it only has solutions for $\epsilon = \hbar\omega(n + 1/2)$, where n is an integer, and $\omega = (k/m)^{1/2}$. In terms of Ginzburg-Landau equations, this means $\kappa^2 = 2H(n+1/2)$, or $H = \kappa/(2n+1)$. The highest field at which solutions exist is $H_{c2} = \kappa$, or, in ordinary units, $H_{c2} = \kappa\sqrt{2}H_{cm}$, where H_{cm} is the critical field in the bulk of superconductor. At higher fields, superconducting state can not exist.