Hilbert space (discrete)

complex vector space w/ inner product

$|\psi_i\rangle$ valid $\Rightarrow$ $\sum_i c_i |\psi_i\rangle$

(c_i ↑ complex)

$\langle \psi_i | \psi_j \rangle = \delta_{ij}$ orthonormal basis

$\forall |\psi\rangle, \ \exists c_i : |\psi\rangle = \sum c_i |\psi_i\rangle$

Conditions on c_i: * $p(i, \text{meas}) = |\langle \psi_i | \psi \rangle|^2 = |c_i|^2$

$\sum p(i) = 1 \Rightarrow$ $\sum |c_i|^2 = 1$

* $|\langle \psi_i | e^{i\phi} | \psi \rangle|^2 = |\langle \psi_i | \psi \rangle|^2 \underbrace{|e^{i\phi}|^2}_{=1}$

$e^{i\phi}|\psi\rangle$ indistinguishable $|\psi\rangle$

overall phase is irrelevant

$a|\chi\rangle + b|\psi\rangle$ vs $a|\chi\rangle + e^{i\phi} b|\psi\rangle$ ARE different!

relative phase

$e^{i\phi_1}|\chi\rangle + e^{i\phi_2}|\psi\rangle$ vs $|\chi\rangle + e^{i(\phi_2-\phi_1)}|\psi\rangle$ SAME

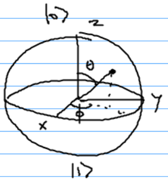
Bloch sphere

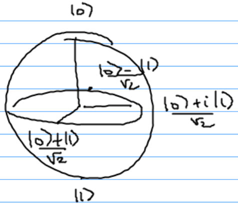
$a|0\rangle + b|1\rangle$

$|a|^2 + |b|^2 = 1$

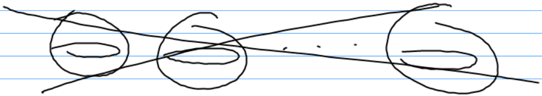
overall phase

$\longrightarrow$ $a|0\rangle + b|1\rangle \iff \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$





More qubits



$|\psi\rangle \otimes |\chi\rangle$; $|\psi\rangle|\chi\rangle$; $|\psi\chi\rangle$

2 qubits: $c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$

$\Sigma |c_i|^2 = 1$, (overall phase)

n qubits : $2^n - 1$ complex d.o.f.

Entanglement

$$(a_1|0\rangle + b_1|1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle)$$

$$= \underbrace{a_1 a_2}_{c_{00}}|00\rangle + a_1 b_2|01\rangle + b_1 a_2|10\rangle + b_1 b_2|11\rangle$$

$$\ldots$$

$$\frac{|00\rangle + |01\rangle}{\sqrt{2}} = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\frac{|00\rangle + |01\rangle - |10\rangle - |11\rangle}{2} = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = ~~(0 + i1)(0 - i1)~~$$

~~$|00\rangle - i|01\rangle + i|10\rangle + |11\rangle$~~

IMPOSSIBLE

NOT SEPARABLE

= ENTANGLED

Vector notation

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \rightarrow \quad |\psi\rangle = a|0\rangle + b|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\langle\psi| = [a^* \;\; b^*] = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger$$

Hermitian conjugate

complex conj. of transpose

$$\langle\psi|\psi\rangle = [a^* \;\; b^*]\begin{bmatrix} a \\ b \end{bmatrix} = a^*a + b^*b$$

$$= |a|^2 + |b|^2$$

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \otimes \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 \\ a_1 b_2 \\ b_1 a_2 \\ b_1 b_2 \end{bmatrix} \begin{matrix} \leftarrow c_{00} \\ \leftarrow c_{01} \\ \leftarrow c_{10} \\ \leftarrow c_{11} \end{matrix}$$

Quantum gates

$U|\psi_i\rangle = |\psi_f\rangle$

$$\begin{bmatrix} U_{00} & U_{10} \\ U_{01} & U_{11} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} U_{00}a + U_{10}b \\ U_{01}a + U_{11}b \end{bmatrix}$$

U unitary $\longrightarrow \begin{bmatrix} \equiv \end{bmatrix}^* \begin{bmatrix} \Big[\,\Big] \Big[\,\Big] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow$ columns, rows orthonormal

$U^\dagger U = I$

Notes:
- $U_2 U_1 |\psi\rangle$

  $U_2 U_1 \neq U_1 U_2$
- $U_1 \otimes U_2 \quad (\neq U_2 \otimes U_1)$
- some 4×4 U CANNOT be written in form $U_1 \otimes U_2$

Example gates

$$I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

$$NOT = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ bit flip

phase flip $$\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$$ $a|0\rangle + b|1\rangle \rightarrow a|0\rangle - b|1\rangle$

Hadamard $$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$|0\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} \rightarrow |0\rangle$

$|1\rangle \rightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}} \rightarrow |1\rangle$

$$I \otimes NOT = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \left[\begin{array}{cc|cc} & 1 & & \\ 1 & & \multicolumn{2}{c}{0} \\ \hline & & & 1 \\ \multicolumn{2}{c|}{0} & 1 & \end{array}\right]$$

$$CNOT = \left[\begin{array}{cc|cc} 1 & & & \\ & 1 & \multicolumn{2}{c}{0} \\ \hline & & & 1 \\ \multicolumn{2}{c|}{0} & 1 & \end{array}\right]$$

00 → 00
01 → 01
10 → 11
11 → 10

(first bit: control; second bit: target)