Density matrix (N&C 2.4)

$\rho = |\psi\rangle\langle\psi|$ pure state

$|\psi\rangle = |0\rangle \rightarrow \rho = |0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$|1\rangle \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$

$a|0\rangle + b|1\rangle \quad = \begin{bmatrix} |a|^2 & a^* b \\ a b^* & |b|^2 \end{bmatrix}$

diagonal elem $\rightarrow$ populations

off-diag $\rightarrow$ coherences

Mixed states - statistical mixture

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \begin{bmatrix} 3/4 & 0 \\ 0 & 1/4 \end{bmatrix}$$

Now: $|a\rangle = \begin{bmatrix} \sqrt{3/4} \\ \sqrt{1/4} \end{bmatrix}$ $\qquad |b\rangle = \begin{bmatrix} \sqrt{3/4} \\ -\sqrt{1/4} \end{bmatrix}$

$$\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b|$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{3/4} \\ \sqrt{1/4} \end{bmatrix}\begin{bmatrix} & \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \\ \end{bmatrix}\begin{bmatrix} & \end{bmatrix} = \begin{bmatrix} 3/4 & \\ & 1/4 \end{bmatrix}$$

Decomposition / interpretation of ρ is NOT UNIQUE

ρ describes ALL our knowledge $\to$ the system

Evolution:

$$\rho_f = |\psi_f\rangle\langle\psi_f|$$

$$\rho_{in} = |\psi\rangle\langle\psi| \longrightarrow \rho_f = \left(U|\psi\rangle\right)\left(\langle\psi|U^\dagger\right)$$

$$= U\rho_{in}U^\dagger$$

also for mixed states $\Rightarrow U\rho_{in}U^\dagger$

Mixed vs pure

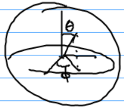
$\mathrm{Tr}(\rho) = 1$ always

$\mathrm{Tr}(\rho^2) = 1 \Leftrightarrow$ pure

$\mathrm{Tr}(\rho^2) < 1 \Leftrightarrow$ mixed

$$\begin{bmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{bmatrix}^2$$

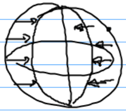
Bloch sphere



pure: on the surface

mixed: inside sphere

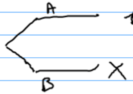
e.g. $\begin{bmatrix} 1/2 & \\ & 1/2 \end{bmatrix} \to (0,0,0)$



Density matrix of subsystem

Reduced den. mat.

Eg. $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$



← what is state of A?

$\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

$\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

$\boxed{\rho_A = \mathrm{Tr}_B\, \rho_{AB}}$

$$\mathrm{Tr}_B\left(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\right) = |a_1\rangle\langle a_2|\ \mathrm{Tr}\,|b_1\rangle\langle b_2|$$
$$= |a_1\rangle\langle a_2|\ \langle b_1|b_2\rangle$$

$$\rho_A = \mathrm{Tr}_B\, \rho_{AB} = \sum_k \langle e_k|\rho|e_k\rangle$$

with $|e_k\rangle$ compl. orthonormal basis of B

$$\frac{1}{2}\Big(\,{}_B\langle 0|\big(|00\rangle + |11\rangle\big)\big(\langle 00| + \langle 11|\big)|0\rangle_B$$
$$+\ {}_B\langle 1|\big(\qquad\big)\big(\qquad\big)|1\rangle_B\Big]$$

$${}_B\langle 0|\ |00\rangle\langle 00|\ |0\rangle_B$$

$$= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{1}{2}\begin{bmatrix}1 & \\ & 1\end{bmatrix} \to \rho_A$$

JOINT STATE ENTANGLED ⇒ COMPONENTS MIXED

$$\rho_B \Rightarrow \left(\begin{array}{cc|cc} X & X & & \\ X & X & & \\ \hline & & X & X \\ & & X & X \end{array}\right)$$

summing 2 submatrices

$$\rho_A \Rightarrow \left(\begin{array}{cc|cc} X & & X & \\ & 0 & & 0 \\ \hline X & & X & \\ & 0 & & 0 \end{array}\right)$$

sum $\begin{matrix} XX \\ XX \end{matrix} + \begin{matrix} 0\,0 \\ 0\,0 \end{matrix}$

$$\frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)$$

$$\rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$\frac{1}{2}\left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 1 \\ & 1 & 1 \end{array}\right)$$

$$\rho_B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\rho_A = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$|1\rangle$

Entangled $\Leftrightarrow$ $\rho_{AB} \neq \rho_A \otimes \rho_B$

Separable $\Leftrightarrow$ $\rho_{AB} = \rho_A \otimes \rho_B$

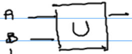
Purification

if mixed ρ_A, can extend system with B

so ρ_{AB} pure (entangled)

$\mathrm{Tr}_B\, \rho_{AB} = \rho_A$

Non-unitary processes 8.2 in N&C



e.g. environment

What happens to A, by itself?

say at $t=0$, $\rho_A \otimes \rho_B$

$$\mathcal{E}(\rho_{A,in}) = \rho_{A,out} = \mathrm{Tr}_B\left[U(\rho_A \otimes \rho_B)U^\dagger\right]$$

say $\rho_B = |e_0\rangle\langle e_0|$

$$\mathcal{E}(\rho_{A,in}) = \sum_k \langle e_k| U(\rho_A \otimes |e_0\rangle\langle e_0|)U^\dagger|e_k\rangle$$

$$= \sum_k \langle e_k|U|e_0\rangle \rho_A \langle e_0|U^\dagger|e_k\rangle$$

$$= \sum_k E_k \rho_A E_k^\dagger$$ → expressed just in fn of A!

Operator sum representation

- E_k need not unitary
- $\sum_k E_k^\dagger E_k = I$ ← trace preserving

$$\left.\begin{bmatrix} \\ \\ \text{superoperator} \\ \\ \\ \end{bmatrix}\begin{bmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{13} \\ \vdots \\ \\ \rho_{44} \end{bmatrix}\right\} \text{16 entries in } 4\times 4\ \rho$$

$$= \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$