# Quantum Measurement

N&C 2.2.3 - 2.2.8

We know: projective measurement in basis $|\psi_m\rangle$

proj. meas. operator $P_m = |\psi_m\rangle\langle\psi_m|$

$P_m P_{m'} = \delta_{ij} P_m$ , $\sum P_m = I$

$p_m = \langle\psi|P_m|\psi\rangle$

post-meas. state $\dfrac{P_m|\psi\rangle}{\sqrt{p(m)}}$

Observable $M = \sum_m m P_m$ → e.g. x-comp of spin 1/2

$E(M) = \sum p_m m$

$= \langle\psi|M|\psi\rangle$

Example : meas. in $|0\rangle, |1\rangle$ basis

meas ops : $P_0 = |0\rangle\langle 0|$ , $P_1 = |1\rangle\langle 1|$

$P_0 + P_1 = I$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$|\psi\rangle = a|0\rangle + b|1\rangle$

$p_0 = \langle\psi|P_0|\psi\rangle = \langle\psi|0\rangle\langle 0|\psi\rangle = |a|^2$

$p_1 = \qquad |b|^2$

post. meas state = $\dfrac{P_0|\psi\rangle}{\sqrt{p_0}} = \dfrac{a|0\rangle}{\sqrt{|a|^2}} = \dfrac{a|0\rangle}{|a|}$

$|0\rangle\langle 0|(a|0\rangle + b|1\rangle)$

## Generalized measurement

$\{\Pi_m\}$ $\qquad p_m = \langle\psi|\Pi_m^\dagger \Pi_m|\psi\rangle$

post-meas state $= \dfrac{\Pi_m|\psi\rangle}{\sqrt{p_m}}$

$\sum p_m = 1 \rightarrow \sum_m \Pi_m^\dagger \Pi_m = I$

① note that $P_m = \Pi_m^\dagger \Pi_m$

$\underline{\Pi_m \Pi_{m'} = \delta_{ij} \Pi_m}$ $\quad \Big\}\ P_m = \Pi_m$

↳ specific to proj. meas.

② $p_m = \mathrm{Tr}(\Pi_m \rho \Pi_m^\dagger) = \mathrm{Tr}(\Pi_m^\dagger \Pi_m \rho)$

post-meas state $\dfrac{\Pi_m \rho \Pi_m^\dagger}{\mathrm{Tr}(\Pi_m^\dagger \Pi_m \rho)}$

POVM Positive operator-valued meas.

$\{E_m\}$

$$\begin{cases} E_m \text{ positive} \\ \sum E_m = I \end{cases}$$

$$p_m = \langle\psi|E_m|\psi\rangle$$

post state don't care

$$\left( \forall E_m,\ \exists M_m : M_m^\dagger M_m = E_m \right)$$

Example POVM

wish to distinguish $|0\rangle$ from $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$

row

$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle\langle 1| \neq \Pi_m$ (matrix)

$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0| - \langle 1|}{\sqrt{2}}\right)$

$E_3 = I - E_1 - E_2$

Suppose you're given a particle and get

outcome 1 $\rightarrow$ NOT $|0\rangle$, IS $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$

2 $\rightarrow$ $|0\rangle$

3 $\rightarrow$ don't know

NEVER make a mistake !

Can we realize such a meas? — YES

Distinguishing q. states (with certainty)

- orthog. states $|\psi_i\rangle$ $\rightarrow$ YES, $\Pi_i = |\psi_i\rangle\langle\psi_i|$

- non-orthog $\rightarrow$ NO

$|\psi_1\rangle$, $|\psi_2\rangle$ non-orthog.

$\rightarrow$ $|\psi_2\rangle = a|\psi_1\rangle + b|\psi_{\perp,1}\rangle$

if measure, can obtain outcome 1
both starting from $|\psi_1\rangle$ or $|\psi_2\rangle$

If you could, you could communicate faster than c

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{(|0\rangle - |1\rangle)(|0\rangle+|1\rangle) - (|0\rangle+|1\rangle)(|0\rangle - |1\rangle)}{2}$$

A could measure in either basis,
if she meas in $|0\rangle, |1\rangle$ basis $\Rightarrow$ B' state will $|0\rangle$ or $|1\rangle$
$|0\rangle \pm |1\rangle$ " $\rightarrow$ $|0\rangle \pm |1\rangle$

## State tomography

one meas → learn very little

if 0 → state is NOT $|1\rangle$

repeated meas in same basis → $|a|^2$, $|b|^2$

learn diag elem of $\rho$

$$\rho = c_0 I + c_x \sigma_x + c_y \sigma_y + c_z \sigma_z$$

diag : $c_0, c_z$

rotate either meas basis OR qubit

$$p_m = \mathrm{Tr}(P_m \rho)$$

other basis $\mathrm{Tr}(\underbrace{U P_m U^\dagger}_{\text{rotated basis}} \rho) = \mathrm{Tr}(P_m \underbrace{U^\dagger \rho U}_{\text{rotate state}})$

Also: Process tomography

$$\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} & \\ & \end{bmatrix} + \frac{1}{4}\begin{bmatrix} & \\ & \end{bmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

00
01
10
11