Feynman (1982)

- universal
- local interconnects
- simulate q. phys.
- allow discretize
- # elem. steps ~ space-time system

Class. deterministic → needs 2^n elem. (size n)

probabilistic → ~ 2^n to find 2^n ampl.

↘ ? imitate prob.

↳ not the same

amplitudes interfere

prob. add up.

Quantum → conjecture: it can be done
imitate

↘ still cannot get all ampl. at end

Notes: • no access to intermediate states

• not all H can be simulated efficiently

Want: $|\psi(t)\rangle = e^{-i\mathcal{H}t}|\psi(0)\rangle$

↳ N particles

- exponentiating $\mathcal{H}$ is expon. hard
- 1st order approx: $|\psi(t+\Delta t)\rangle = (I - i\mathcal{H}\Delta t)\,|\psi(t)\rangle$

  efficient, but crude
- $\mathcal{H} = \sum_k \mathcal{H}_k$

  ↳ acts on at most m particles

  poly(N)

$e^{-i\mathcal{H}_k \Delta t}$ can be simulated efficiently

$[\mathcal{H}_j, \mathcal{H}_k] \neq 0$

$$e^{-i\mathcal{H}t} \neq \prod_k e^{-i\mathcal{H}_k t}$$

$$\lim_{n\to\infty}\left(e^{iAt/n}\,e^{iBt/n}\right)^n = e^{i(A+B)t}$$ Trotter formula

$$e^{iAt/n} = I + iA\frac{t}{n} + O\left(\frac{1}{n^2}\right)$$

$$e^{iAt/n}\,e^{iBt/n} = I + i(A+B)\frac{t}{n} + O\left(\frac{1}{n^2}\right)$$

$$= e^{i(A+B)t/n} + O\left(\frac{1}{n^2}\right)$$

$$\left(\qquad\qquad\right)^n = e^{i(A+B)t} + O\left(\frac{1}{n}\right) \qquad (*)$$

choose $n > \frac{1}{\varepsilon} \to$ error

$$e^{\frac{A}{2}\Delta}\,e^{B\Delta}\,e^{\frac{A}{2}\Delta} \to \sim \Delta^3$$

$$e^{A\Delta}\,e^{B\Delta} \to \sim \Delta^2$$

Put things together: (Lloyd 96)

- $e^{i \mathcal{H}_k t/n}$ takes $\leq m^2$ ($\mathcal{H}_k$ act on m)
- $\mathcal{H} = \sum^{L} \mathcal{H}_k \rightarrow \left(\prod^{L} e^{i \mathcal{H}_k t/n} \right)^n$

$\hookrightarrow L \cdot n$

$\rightarrow \boxed{n L m^2}$

$\downarrow$

rep.

(*) error $E(U,V)$

$$E(U_m U_{m-1} \cdots U_1,\ V_m V_{m-1} \cdots V_1) \leq \sum_j E(U_j, V_j)$$

def. $E(U_1, V_1) = \| (U_1 - V_1)|\psi\rangle \|_{\max |\psi\rangle}$

$$E(U_2 U_1, V_2 V_1) = \| (U_2 U_1 - V_2 V_1)|\psi\rangle \|$$

$$= \| (U_2 U_1 - V_2 U_1)|\psi\rangle + (V_2 U_1 - V_2 V_1)|\psi\rangle \|$$

$$\leq \| (\qquad)|\psi\rangle \| + \| (\qquad)|\psi\rangle \|$$

$$\leq E(U_2, V_2) + E(U_1, V_1)$$

1D particle (Wiesner, Zalka)

$H = \frac{p^2}{2m} + V(x)$

$|\psi\rangle = \int |x\rangle\langle x|\psi\rangle\, dx \approx \sum_{-d/\Delta x}^{d/\Delta x} a_k\, |k\,\Delta x\rangle$

$\left\lceil \log_2\left(\frac{2d}{\Delta x} + 1\right)\right\rceil$ qubits

$[\frac{p^2}{2m}, V] \neq 0$ in general

- $e^{-iV(k\Delta x)\Delta t}$ is diagonal in x

  $\rightarrow$ easy if $V(x)$ easily computable

- $e^{-i\frac{p^2}{2m}\Delta t} = U_{QFT}\, e^{-\frac{ix^2}{2m}\Delta t}\, U^\dagger_{QFT}$

  $\hookrightarrow$ also diag in x-space

Finding eigenvalues, eigenvectors of $\mathcal{H}$

$|\psi\rangle = |0\rangle|v_a\rangle$ $\hookrightarrow$ approx. eigenstate of U

$\xrightarrow{H^{\otimes n}} \quad = \sum_j |j\rangle |v_a\rangle$

$\rightarrow \quad = \sum |j\rangle U^j |v_a\rangle$ $\quad$ $\rightarrow$ actual eigenstates of U

$= \sum_j |j\rangle U^j \sum_k c_k |v_k\rangle$

$= \sum_k c_k \sum_j |j\rangle \underbrace{U^j |v_k\rangle}_{\lambda_k^j |v_k\rangle}$ $\qquad U|v_k\rangle = \lambda_k |v_k\rangle$

$\lambda_k = e^{i\omega_k}$

$= \sum_k c_k |v_k\rangle \underbrace{\sum_j e^{i\omega_k j}|j\rangle}_{\omega_k N}$

$\xrightarrow{QFT^\dagger}$

$QFT: |x\rangle \rightarrow \sum_y e^{2\pi i \frac{x}{N} y} |y\rangle$

Errata N&C:

http://squint.org/cgi