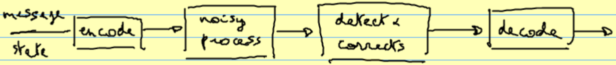
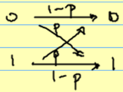
QUANTUM ERROR CORRECTION

Reliable q. comp. with unreliable components

How: encoding, correcting, decoding



Classical example



encoding

$0_L \to 000$

$1_L \to 111$

decoding

majority vote

$P(\text{failure}) = p$ w/o encoding

$= 3p^2(1-p) + p^3$ with encoding

Encoding helps if $3p^2(1-p) + p^3 < p$

$p < 1/2$

Is QEC possible? Why (maybe) not?

1. no cloning - theorem
2. errors are continuous
3. measuring destroys state

1995: Shor, Steane

1. entanglement
2. digitize errors
3. measure error

3 qubits bit-flip errors

$$|\psi\rangle \xrightarrow{1-p} |\psi\rangle$$
$$\searrow_p \; X|\psi\rangle$$

- encode $|0\rangle \to |000\rangle$, $|1\rangle \to |111\rangle$

  $a|0\rangle + b|1\rangle \to a|000\rangle + b|111\rangle$ NOT a copy

  copy: $(a|0\rangle + b|1\rangle) \otimes (\quad) \otimes (\quad)$

- measure error (syndrome)
  - comparison of q. 1 & 2 : $Z_1 Z_2 = (|00\rangle\langle 00| + |11\rangle\langle 11| - |01\rangle\langle 01| - |10\rangle\langle 10|) \otimes I$

    outcome 1 → equal

    0 → not equal

    $a|010\rangle + b|101\rangle$

  - comparison of q. 2 & 3 : $Z_2 Z_3 = \ldots$

- correct

$Z_1 Z_2$	$Z_2 Z_3$	action
1	1	I
1	0	X_3
0	1	X_1
0	0	X_2

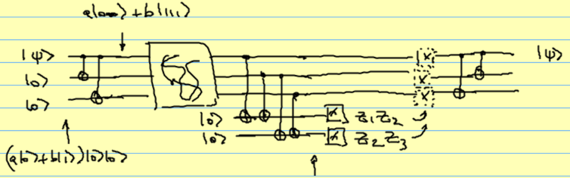
$$a|010\rangle + b|101\rangle \xrightarrow{Z_1 Z_2} -a|010\rangle - b|101\rangle$$

outcome -1

$$\xrightarrow{Z_2 Z_3} a|010\rangle + b|101\rangle$$

outcome $+1$

How do we do this?



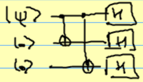
3 qubit phase flip code

$$|\psi\rangle \xrightarrow{1-p} |\psi\rangle$$
$$\searrow_{p}\ Z|\psi\rangle$$

Note : $HZH = X$

Method ① insert H before & after error process
use bit flip code

② encode



$|0\rangle_L \to (|0\rangle+|1\rangle)(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)$

$|1\rangle_L \to$ − − −

$|0\rangle_L = |+\rangle|+\rangle|+\rangle$

$|1\rangle_L = |-\rangle|-\rangle|-\rangle$

detecting X_1X_2 etc

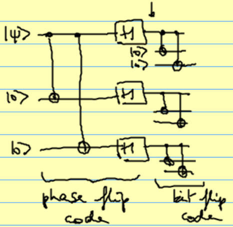
correcting Z_1, Z_2, Z_3

decoding : inverse of encoding

9-qubit code for arbitrary errors

$|0\rangle_L = (|000\rangle + |111\rangle)(\quad + \quad)(\quad + \quad)$

$|1\rangle_L = (|000\rangle - |111\rangle)(\quad - \quad)(\quad - \quad)$



bit flip error det.: $Z_1Z_2,\ Z_2Z_3$

$Z_4Z_5,\ Z_5Z_6$

$Z_7Z_8,\ Z_8Z_9$

phase flip: $X_1X_2X_3X_4X_5X_6,$

$X_4X_5X_6X_7X_8X_9$

$-iY = XZ$

Arbitrary errors

$$E_i = e_{i0} I + e_{i1} X + e_{i2} Y + e_{i3} Z$$

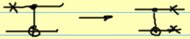
$$\rho \to \sum_i E_i \rho E_i^\dagger$$

measure syndrome collapses state into one of $E_i \rho E_i^\dagger$
outcome gives corresponding information

$$\rho + X_1 \rho X_1 + X_2 \rho X_2 + X_3 \rho X_3$$

Fault-tolerant QC - QC with encoded states

Problems: ① errors can happen in state prep., gates, meas. ...

②  errors can propagate

Idea: avoid ≥ 2 errors within 1 code block
(1 error is ok, correctable)

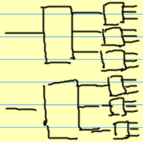
→ count all poss. errors

$$\underset{\text{in 1 block}}{p(\geq 2\,\text{errors})} = c\,p^2$$

improvement $c p^2 < p$

$$p \leq p_{th} = c^{-1} \approx 10^{-4}$$

Concatenation



...

For 1 actual comp. step

	P(failure)	# gates
no encoding	p	1
1 layer	cp^2	d
2 layers	$c(cp^2)^2$	d^2
⋮		
k	$\frac{(cp)^{2^k}}{c}$	d^k

expon small error with poly effort