The original game in characteristic function form is:

$v(\phi)=0$
$v(A)=21 \quad v(B)=18 \quad v(C)=8.5$
$v(AB)=60 \quad v(AC)=40 \quad v(BC)=37$
$v(ABC)=100$

We seek the game in its strategic equivalent form

$v(\phi)=0$
$v(A)=21-v(A) \quad v(B)=18-v(B) \quad v(C)=8.5-v(C)$
$v(AB)=60-v(A)-v(B) \quad v(AC)=40-v(A)-v(C) \quad v(BC)=37-v(B)-v(C)$
$v(ABC)=100-v(A)-v(B)-v(C)$

$v(\phi)=v(A)=v(B)=v(C)=0$
$v(AB)=21 \quad v(AC)=v(BC)=10.5$
$v(ABC)=52.5$

$v(\phi)=v(A)=v(B)=v(C)=0$
$v(AB)=21/v(ABC) \quad v(AC)=v(BC)=10.5/v(ABC)$
$v(ABC)=52.5/v(ABC)$

$v(\phi)=v(A)=v(B)=v(C)=0$
$v(AB)=0.40 \quad v(AC)=v(BC)=0.20$
$v(ABC)=1.00$

SOLUTION NO. ONE: THE CORE

$X_C<.60$ since $v(AB)=0.4$
$X_A<.80$ since $v(BC)=0.2$
$X_B<.80$ since $v(AC)=0.2$
$X_A+X_B+X_C=1.0$ since $v(ABC)=1$

See figure #1, the Core where this is graphed. I would use the Core if I wanted to find a solution which is in everyone's best interest.

## SOLUTION NO. TWO: THE STABLE SET

This game, like the "Split the Dollar" game, permits a symmetric stable solution.* There are also a range of "skewed" solutions inside the core.

Let

$$0 > \alpha \geq 0.60$$

where $\alpha$ is the share to player C. Players A and B may split the remainder so long as the imputation remains within the core (Figure #3).

You would use the stable set to represent collusive bargaining processes.

* See figure 2.

## SOLUTION NO. THREE: THE SHAPLEY VALUE

A table of marginal values for the Shapley value

| | A | B | C |
|---|---|---|---|
| ABC | 0 | 0.40 | 0.60 |
| ACB | 0 | 0.80 | ~~0.80~~ 0.20 |
| BAC | 0.40 | 0 | 0.60 |
| BCA | 0.80 | 0 | 0.20 |
| CAB | 0.20 | 0.80 | 0.0 |
| CBA | 0.80 | 0.20 | 0 |
| total | 2.20 | 2.20 | 1.60 |
| ave | 0.37 | ~~0.27~~ 0.37 | ~~0.37~~ 0.27 |

is given above. You should use the Shapley value to impute a fair value for the game.

# SOLUTION NO. FOUR: THE BARGAINING SET

The partitions are as follows

$\{A\}\{B\}\{C\}$
$\{A\}\{BC\}$
$\{AB\}\{C\}$
$\{AC\}\{B\}$
$\{ABC\}$

We would use the bargaining set to model the situation where who the players knew mattered to the bargain

The imputation for $\{A\}\{B\}\{C\}$ is $\{0,0,0\}$
The imputation for $v\{A\}\{BC\}$ is calculated as follows

$$v\{AB\} - X_B = v\{AC\} - X_C$$
$$.40 - X_B = .20 - X_C$$
$$.20 + X_C = X_B$$
$$\{0, .20, 0\} \qquad v\{A\} = 0 \quad v\{B,C\} = 0.20$$

The imputation for $\{AB\}\{C\}$ is calculated as follows

$$v\{AC\} - X_A = v\{BC\} - X_B$$
$$X_A = X_B$$
$$v\{C\} = 0 \quad v\{AB\} = 0.40$$
$$\{0.20, 0.20, 0\}$$

The imputation for $\{B\}\{AC\}$ is calculated as follows

$$v\{AB\} - X_A = v\{CB\} - X_C$$
$$0.40 - X_A = v\{CB\} - X_C$$
$$0.4 - X_A = 0.20 - X_C$$
$$0.2 + X_C = X_A$$
$$v\{AC\} = 0.2$$
$$\{0.2, 0, 0\}$$

The imputation for $\{ABC\}$ is calculated as follows

$$3X + 40 = 100$$
$$3X = 60$$
$$X = \cancel{26.7}\ 20$$

based upon the fixed ratios above

~~$\{27, 27, 47\}$~~
$\{.40, .40, .20\}$

# SOLUTION NO. FIVE: THE NUCLEOLUS

See Figure 4. We must first deal with the excess of coalitions {BC} and {AC} and by implication of {A} and {B} as well. The clear solution involves centering the point midway between apex A and B. This leaves the excess of {AB} and {C}. We must also center these two coalitions, resulting in the imputation $X_A = .35$, $X_B = .35$, $X_C = .30$. The excess of the one player coalitions are

$$e_A = v\{A\} - .35 = -.35$$
$$e_B = v\{B\} - .35 = -.35$$
$$e_C = v\{C\} - .30 = -.30$$

The excess of the two player coalitions are

$$e_{AB} = v\{AB\} - .40 = -.30$$
$$e_{AC} = v\{AC\} - .20 = -.45$$
$$e_{BC} = v\{BC\} - .20 = -.45$$

We would use the nucleolus if we wanted to minimize the unhappiness withe the solution.

# SOLUTION NO. SIX: THE GATELY POINT

The ratio of values for the Gately point would be the following

$$\gamma = v\{ABC\} - v\{A\} : v\{ABC\} - v\{B\} : v\{ABC\} - v\{C\}$$

$$\gamma = x_A : x_B : x_C$$

$$= 80 : 80 : 60$$

$$= 8 : 8 : 6$$

$$\gamma = \{36, 36, 27\}$$

I would use the Gately point to minimize the disruption to the coalition

## COMPARISON

The six solution concepts are comparable. The stand-out is the bargaining set, which predicts that A and B will conspire against C. The symmetric stable set is also quite distinct since it presumes collusion on the part of two players, thereby excluding the third. The symmetry between players A and B were respected by all solution concepts.

when we reverse the solution back to native units we must multiply everything by 52.5, and add $u_c = (21, 18, 8.5)$ to the result

The Core converted

$$\hat{x}_C < 79$$
$$\hat{x}_B < 63$$
$$\hat{x}_A < 91.5$$
$$\hat{x}_A + \hat{x}_B + \hat{x}_C = 100$$

The symmetric stable set converted

$$\hat{x} = \{50, 50, 0\}$$
$$= \{50, 0, 50\}$$
$$= \{0, 50, 50\}$$

Shapley value converted

$$\hat{x} = \{40, 37, 23\}$$

Bargaining set converted

$$\hat{x} = \{42, 39, 19\}$$

Nucleolus

$$\hat{x} = \cancel{\{36, 30, 24\}}$$
$$\{39, 36, 24\}$$

Gately Point

$$\hat{x} = \{40, 37, 23\}$$

The symmetries of the problem as originally stated were not immediately apparent.

Figure #1. The Cone

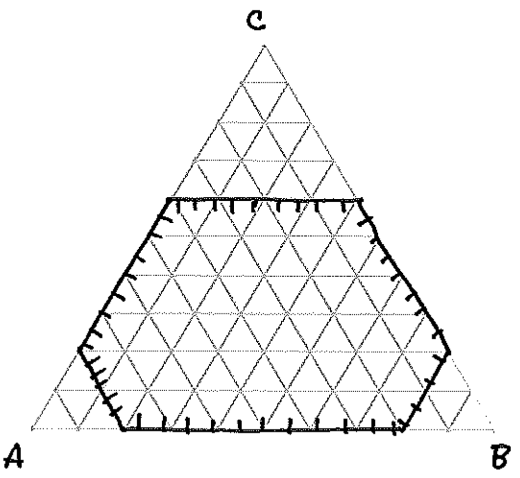


Figure #2. Symmetric Stable Set Solution

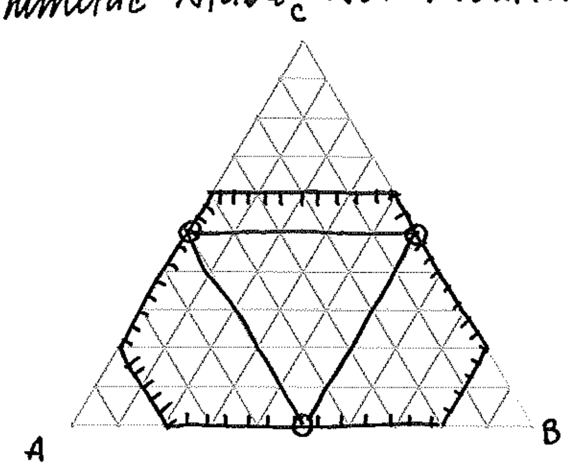


Figure #3. Secured Stable Set Solution

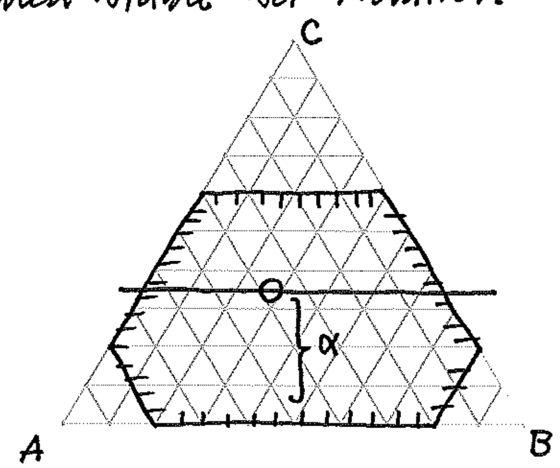

Figure #4: The Nucleolus

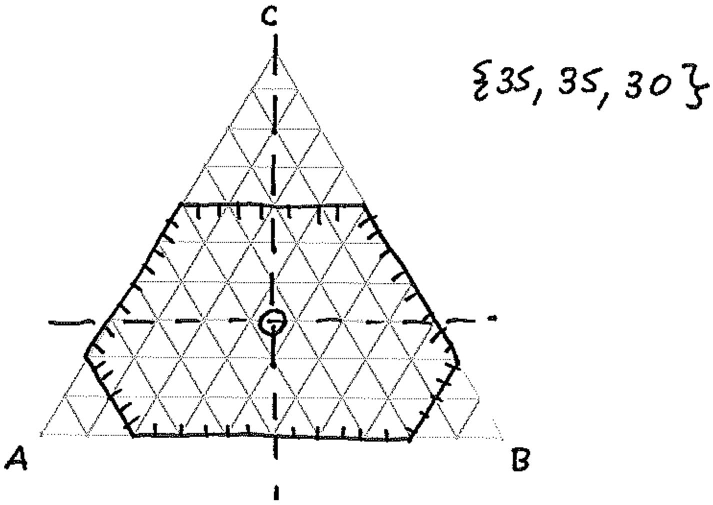


Figure #5 Comparison and Contrast