3 Underwater propagation

3.2 Ray acoustics

3.2.1 Relevant mathematics

We first consider a plane wave as depicted in figure 1. As shown in the figure wave fronts are planes. The arrow perpendicular to the wave front indicates the direction in which the acoustic energy flows. The arrow is called a (sound) 'ray'. A few additional rays are indicated in the figure (dashed arrows). By convention the intensity is inversely proportional to the distance between the rays. Then for a plane wave rays should be depicted equidistantly.

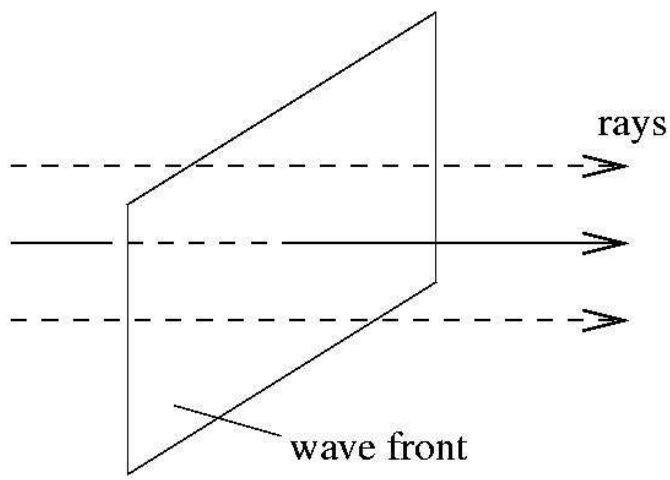

Figure 1

Next we consider a spherical wave due to a point source in a homogeneous medium (see figure 2a). Wave fronts are spheres. Perpendicular to the wave fronts a few rays have been drawn forming a so-called ray diagram. The number of rays per unit area is proportional to the sound intensity. Hence, at a distance r from the point source intensity is inversely proportional to the area of a sphere with radius r, i.e., $1/r^2$. Now consider the medium to be inhomogeneous, e.g., sound speed decreases linearly with increasing depth. The propagation wave now changes with direction resulting in non-spherical wave fronts (see figure 2b). The sound rays, still perpendicular to the wave fronts, now follow curved paths with a varying radius of curvature. This is called 'refraction' of sound rays. Still, the local density of sound rays is measure for the local sound intensity.

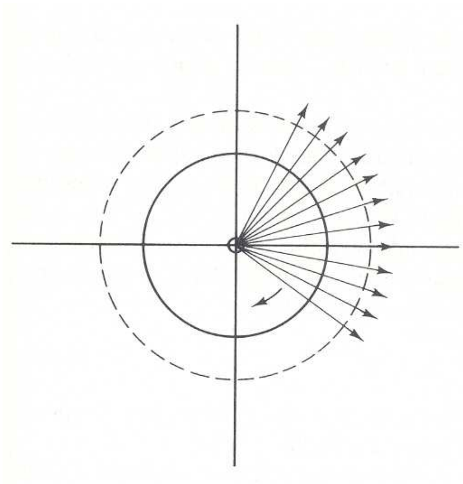

Figure 2a

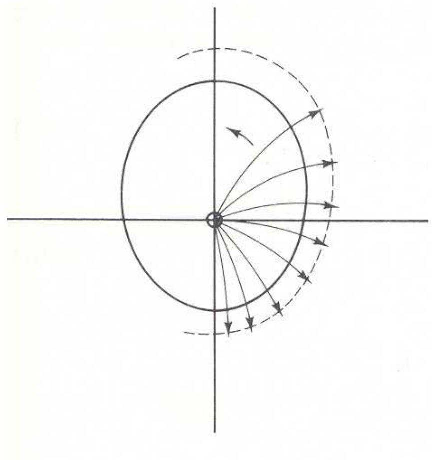

Figure 2b

We will now treat the mathematics of the refraction of a sound ray in a medium (the ocean) where sound speed varies linearly with depth z:

$$c(z) = c_0 + g\,z \tag{1}$$

with

c_0 the sound speed at a reference depth (we will take z = 0 m) en g the sound speed gradient (unit: s^{-1}).
Snell's law applies along the sound ray

$$\frac{\cos\theta(z)}{c(z)} = \text{constant}$$

or

$$\frac{\cos\theta(z)}{c(z)} = \frac{\cos\theta_0}{c_0} = \frac{1}{c_0} \tag{2}$$

We choose $\theta_0 = 0$, i.e., $z = 0$ is the depth where the sound ray is horizontal. At this depth $z = 0$ the speed of sound equals c_0.

We differentiate equations (1) and (2) with respect to depth z, yielding:

$$dc = g\,dz \quad \text{and} \quad dc = -c_0 \sin\theta\, d\theta$$

Hence

$$dz = \frac{-c_0 \sin\theta}{g}\, d\theta \tag{3}$$

We compare this result with the motion of a point mass on a circle with radius *R*, see figure 3.

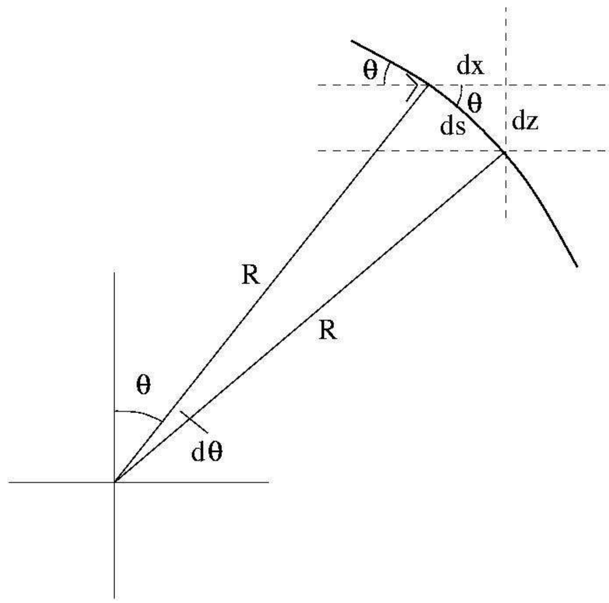

Figure 3

We derive

$$\sin\theta = \frac{dz}{ds} = \frac{dz}{R\,d\theta}$$

or

$$dz = R\sin\theta\, d\theta \qquad (4)$$

Comparing equation (4) with equation (3) we obtain

$$\boxed{R = -\frac{c_0}{g} = -\frac{c(z)}{g\cos\theta(z)}} \qquad (5)$$

- when $g < 0$ then $R > 0$, i.e., downward refraction of the ray;
- when $g > 0$ then $R < 0$, i.e., upward refraction of the ray.

From figure 3 we also derive $dx = R\cos\theta\, d\theta$.
We will derive a few additional formulas for the circular motion, see figure 4. We consider the situation $g < 0$.

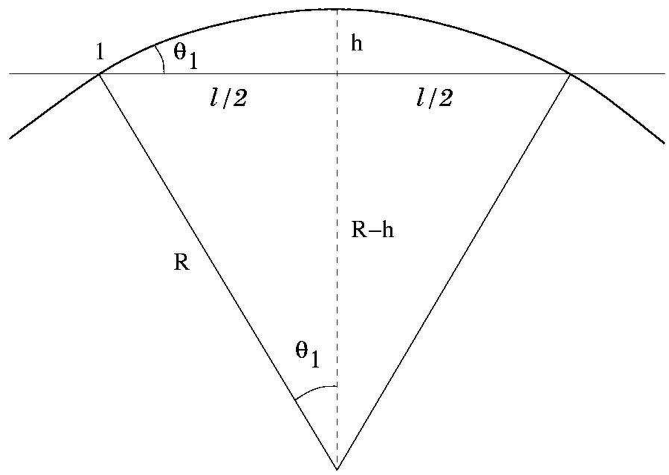

Figure 4

At position 1 the angle with the horizontal and the sound speed is denoted θ_1 and c_1, respectively. The maximum height h is given by

$$\cos\theta_1 = \frac{R-h}{R}$$

Hence

$$h = R\left(1-\cos\theta_1\right) \tag{6}$$

with

$$R = -\frac{c_1}{g\cos\theta_1}$$

The chord ℓ is derived from

$$\sin\theta_1 = \frac{\ell}{2R}.$$

Hence

$$\ell = 2R\sin\theta_1 \tag{7}$$

Next we consider two arbitrary points 2 and 3 on the circle as depicted in figure 5 below.

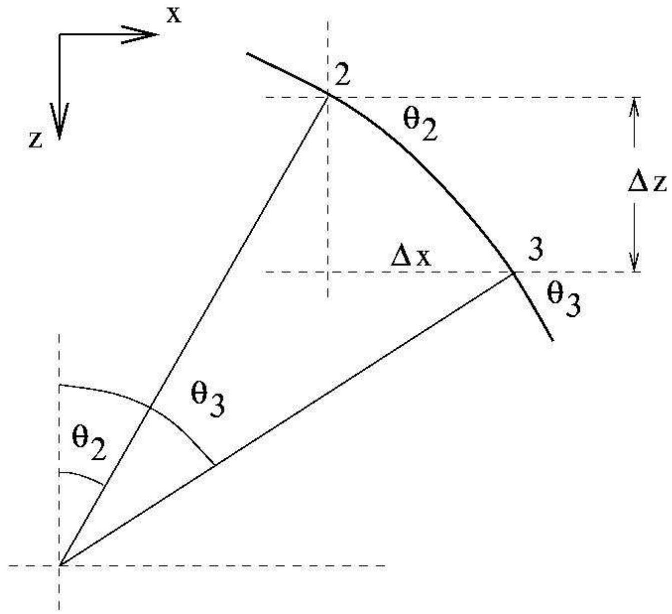

Figure 5

We already derived $dz = R\sin\theta\, d\theta$ and $dx = R\cos\theta\, d\theta$.
Hence

$$\Delta z = \int_{\theta_2}^{\theta_3} dz = \int_{\theta_2}^{\theta_3} R\sin\theta\, d\theta = R\left(\cos\theta_2 - \cos\theta_3\right) \tag{8}$$

and

$$\Delta x = \int_{\theta_2}^{\theta_3} dx = \int_{\theta_2}^{\theta_3} R\cos\theta\, d\theta = R\left(\sin\theta_3 - \sin\theta_2\right) \tag{9}$$

From $R = -\dfrac{c_i}{g\cos\theta_i}$ we also derive

$$\Delta z = \frac{c_3 - c_2}{g} \tag{10}$$

3.2.2 An example ray calculation

We consider the bilinear sound speed profile as given in figure 6.

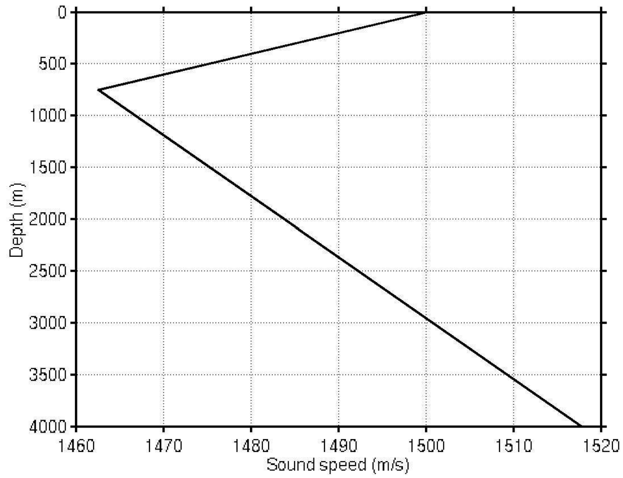

Figure 6

The gradient in the upper layer is g_0 = -0.05 s^{-1} and in the second layer g_1 = +0.017 s^{-1}. The kink in the profile is a depth of 750 m. The value of g_1 agrees with the hydrostatic pressure term in the empirical equations for the sound speed as a function of temperature, salinity and depth.
An omni-directional (point) source is positioned at a depth of 100 m. We compute the sound ray launched at the source with angle θ_0 = 0°. The rays consists of consecutive parts of circles with different radius of curvatures, see figure 7.

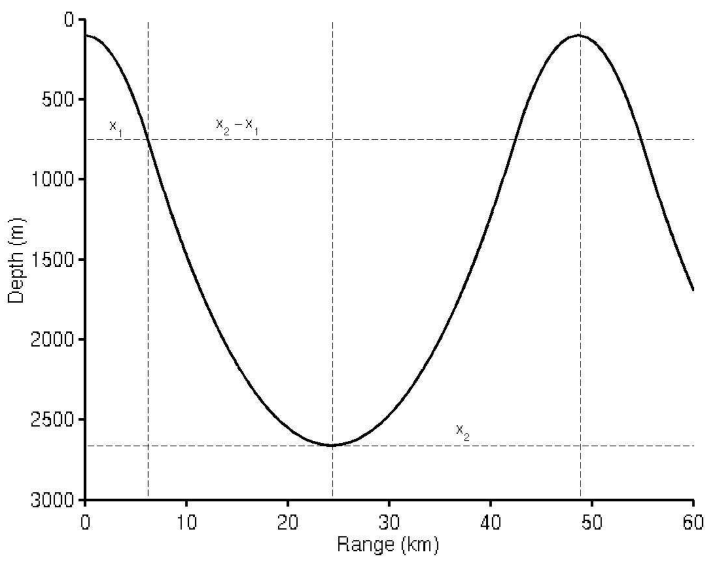

Figure 7

- Sound speed at z = 100 m: c_0 = 1500 + 100 g_0 = 1495 m/s
- Radius of curvature of the ray up to z = 750 m: $R_0 = -\dfrac{c_0}{g_0}$ = 29900 m

- Angle θ_1 with the horizontal at z = 750 m (Snell's law): $\frac{\cos\theta_1}{c_1} = \frac{\cos\theta_0}{c_0}$
  with c_1 = 1500 +750 g_0 = 1462.5 m/s: θ_1 = 11.97°.

- Horizontal distance x_1 up to z = 750 m: $x_1 = R_0(\sin\theta_1 - \sin\theta_0) = 6201\,\text{m}$

- Radius of curvature of the sound ray below z = 750 m (i.e. in the second layer):

$$R_1 = -\frac{c_1}{g_1 \cos\theta_1} = -87941\,\text{m}$$

- Depth h (with respect to z = 750 m): $h = R_1(1-\cos\theta_1) = 1912$ m.
  The minimum depth of the ray is thus z_2 = 750 + h = 2662 m.

  We can also calculate z_2 in a different way:

  $\frac{\cos\theta_2}{c_2} = \frac{\cos\theta_0}{c_0}$ with $\theta_2 = \theta_0 = 0°$ and thus $c_2 = c_0$ = 1495 m/s
  Now $c_2 = c_1 + g_1(z_2 - 750)$, from which it follows z_2 = 2662 m.

- Horizontal distance $(x_2 - x_1)$ up to $z = z_2$:

$$x_2 - x_1 = R_1(\sin\theta_2 - \sin\theta_1) = 18237\text{ m}$$

  Hence x_2 = 24438 m.

Remarks:

- The sound ray has completed a complete 'cycle' after a horizontal distance $2x_2$ (after which the cycle repeats);
- The sound ray calculations given above hold for all horizontal directions.

3.2.3 Propagation loss

In the previous paragraph we have computed one sound ray. In order to get an impression of the complete sound intensity field as a function of range x and depth z we have to compute many sound rays for consecutive values for the launch angle at the source. We will thereby obtain a ray diagram (the difference $\Delta\theta$ in launch angle for two consecutive rays is usually taken fairly small, i.e., a fraction of a degree). This procedure is called 'ray tracing'. Again the intensity of the sound field at some distance and depth follows from the local density of sound rays.
We consider two rays with launch angles $\theta_1 + \Delta\theta/2$ and $\theta_1 - \Delta\theta/2$ with the horizontal (figure 8).

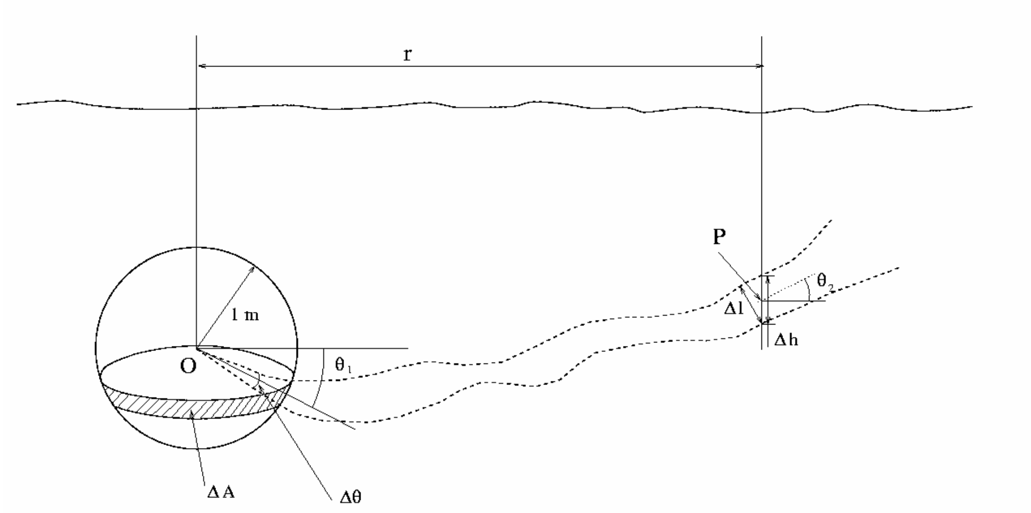

Figure 8

At position P (horizontal range *r*, depth *z*) the angle with the horizontal is θ_2 for both rays. The vertical distance between the rays at P is Δh.

Assume the total radiated energy by the source is ΔP. The energy within $\Delta\theta$ cannot 'leak out' of the tube formed by the pair of rays. The intensity at 1 m from the source in the direction θ_1 is

$$I_1 = \frac{\Delta P}{\Delta A_1} \tag{11}$$

with ΔA_1 the area on the sphere (with 1 m radius) subtended by the two rays (indicated by the shaded area in figure 8).
At P the sound intensity is given by

$$I_2 = \frac{\Delta P}{\Delta A_2} \tag{12}$$

with ΔA_2 the area included by the two rays in the vicinity of P, taken perpendicular to the direction θ_2. Hence, the propagation loss (also called transmission loss) at P is

$$PL = 10^{10}\log\frac{I_1}{I_2} = 10^{10}\log\frac{\Delta A_2}{\Delta A_1} \quad \text{[dB]} \tag{13}$$

with

$$\Delta A_1 = 2\pi\cos\theta_1\Delta\theta \tag{14}$$

whereas

$$\Delta A_2 = 2\pi r \Delta l = 2\pi r \Delta h \cos\theta_2 \qquad (15)$$

r being the horizontal distance between the source position O and P and Δl the 'perpendicular distance' between the two rays at P. Δh is the vertical separation between the two rays at P. The expression for *PL* becomes

$$PL = 10^{10}\log\frac{r\Delta h\cos\theta_2}{\Delta\theta\cos\theta_1} \qquad (16)$$

Using Snell's law $\left(\frac{\cos\theta_2}{\cos\theta_1} = \frac{c_2}{c_1}\right)$ we obtain

$$PL(r,z) = 10^{10}\log\left(\frac{r\Delta h\, c_2}{\Delta\theta\; c_1}\right) \approx 10^{10}\log\left(\frac{r\Delta h}{\Delta\theta}\right) \qquad (17)$$

where the ratio $\frac{c_2}{c_1}$ can be approximated by unity because of the relatively small differences in sound speed in the water column. (*r*,*z*) is the position of P.

<u>Remark:</u>
In a homogeneous medium (no refraction) equation (17) reduces to

$$PL = 20\ {}^{10}\log r \qquad (18)$$

because then $\Delta h = r\,\Delta\theta$ ('spherical spreading').

For the sound speed profile of the previous paragraph we provide the results of the computations for a second sound ray with launch angle θ_0 = -4° at the source:

R_0 = 29973 m
θ_1 = 12.61°
x_1 = 4453 m
R_1 = -88156 m
h = 2127 m
z_2 = 2877 m
x_2 = 23699 m

Both rays are depicted in figure 9 below.

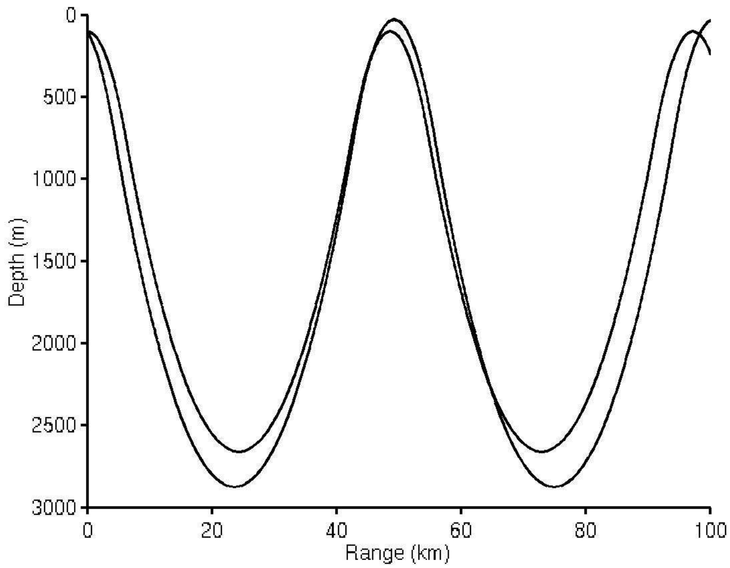

Figure 9

This figure shows that the two rays converge and diverge, i.e., there are large deviations from spherical spreading. We also notice that the rays cross. According to equation (17) the sound intensity should be infinite at these intersections. As this is physically impossible, more rigorous solutions of the wave equation are needed for these situations.

Still, ray diagrams can provide useful insight.

Finally, we remark that besides refraction PL is also enormously influenced by interaction of the sound with the rough sea surface and the rough and layered ocean bottom.

Hence, sophisticated computer models have been developed for sound propagation in the ocean.

3.2.4 Case studies

Deep water with a negative sound speed gradient below sea surface

We consider a deep water situation (water depth H = 4000 m) where the upper part of the sound speed profile (i.e. below the sea surface) exhibits a negative constant gradient.

As an example we take $g = -0.05\ s^{-1}$, $c(z = 0) = c_0 = 1500$ m/s.

For a point source at depth z_s = 120 m figure 10 presents the sound rays with launch angles in between -3°and + 6° (angle increment 1°).

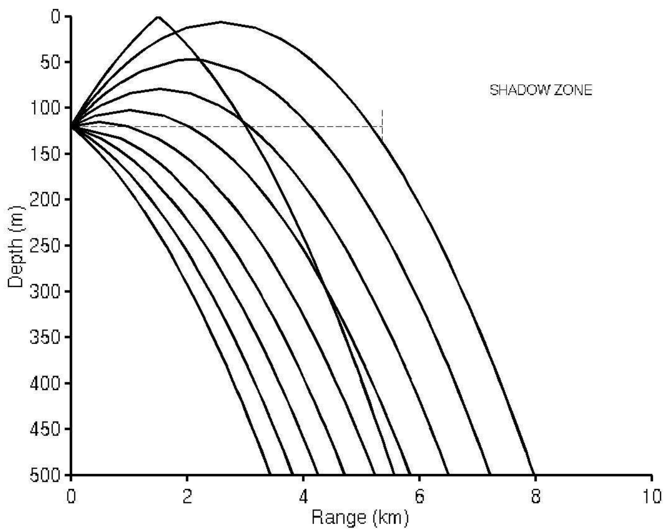

Figure 10

We notice the existence of a sound ray (with launch angle in between 5° en 6°) that just hits the sea surface. Rays with a larger launch angle reflect off the sea surface (such as the ray with a 6° launch angle). This results in a region, indicated in the figure by ‘shadow zone’, where sound rays cannot enter directly. Sound energy can only enter this region via sound rays having a negative launch angle that reflect off the sea bottom.
Using equation (6) we calculate the launch angle θ_m of the ray that just hits the sea surface (taking h equal to z_s):

$$z_s = R(1-\cos\theta_m)$$

where the radius of curvature R of that ray is given by

$$R = -\frac{c(z_s)}{g\cos\theta_m} = -\frac{c_0}{g}\,.$$

Hence

$$\theta_m = \arccos\left(1+\frac{z_s g}{c_0}\right) \qquad (19)$$

We now consider a omni-directional (point) receiver at a depth z_r (taken to be the same as that of the source). At this depth the shadow zone begins at a horizontal distance x_m given by

$$x_m = 2R\sin\theta_m \qquad (20)$$

For the numerical values in this example we find θ_m = 5.126° en x_m = 5361 m.

For a source receiver range $r < x_m$ propagation loss can be well approximated by the spherical spreading law (including absorption):

$$PL = 60 + 20\ {}^{10}\log r + \alpha r \ \ [\text{dB}] \tag{21}$$

with r in km and α in dB/km.
For $r > x_m$ PL is determined by sound rays having negative launch angle that reflect off the sea bottom, see figure 11.

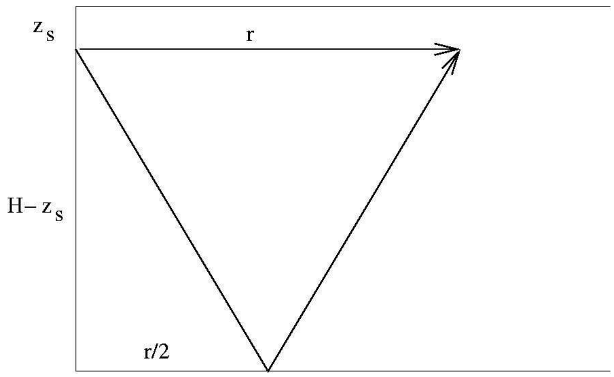

Figure 11

Assuming these sound rays consist of straight lines (by virtue of their relatively large launch angles and thus large radii of curvature), the distance s travelled by such a ray is given by

$$s = 2\sqrt{(H - z_s)^2 + \left(\frac{r}{2}\right)^2} \tag{22}$$

Propagation loss for $r > x_m$ is now given by

$$PL = 60 + 20\ {}^{10}\log s + \alpha s + BL \qquad (s \text{ in km}) \tag{23}$$

This equation contains the spherical spreading law for the distance travelled s and the so-called bottom loss BL (in dB). BL depends on the angle of incidence at the bottom and thus depends on range r.

We will compute PL for the following situation:
H = 4000 m, f = 3000 Hz (α = 0.2 dB/km), $z_s = z_r$ = 120 m and BL = 10.5 (reflection coefficient 0.3). For the sake of simplicity we take BL independent of incidence angle and thus independent of range r. For ranges up to say 15 km this is allowed, since the grazing angle at the bottom is then larger than 27°. BL of a sandy bottom for grazing angles in excess of 27° can be taken approximately constant, since the reflection coefficient quickly approaches a constant value for grazing angles greater than the critical angle.

Figure 12 presents *PL* as a function of range *r* ($0 < r < 15$ km) for the direct sound path and the bottom reflected path separately, both with and without absorption. For the bottom-reflected path we also present *PL* without the *BL* term.

The thick solid lines indicate where the *PL* expressions (21) and (23) are valid:

- for $r < x_m$ *PL* is determined by the direct sound path (apply equation (21));
- for $r > x_m$ *PL* is determined by the bottom-reflected sound path (apply equation (23)).

Since travelled distance *s* changes only slowly with range *r*, *PL* for the bottom-reflected path is a much flatter function of *r*.

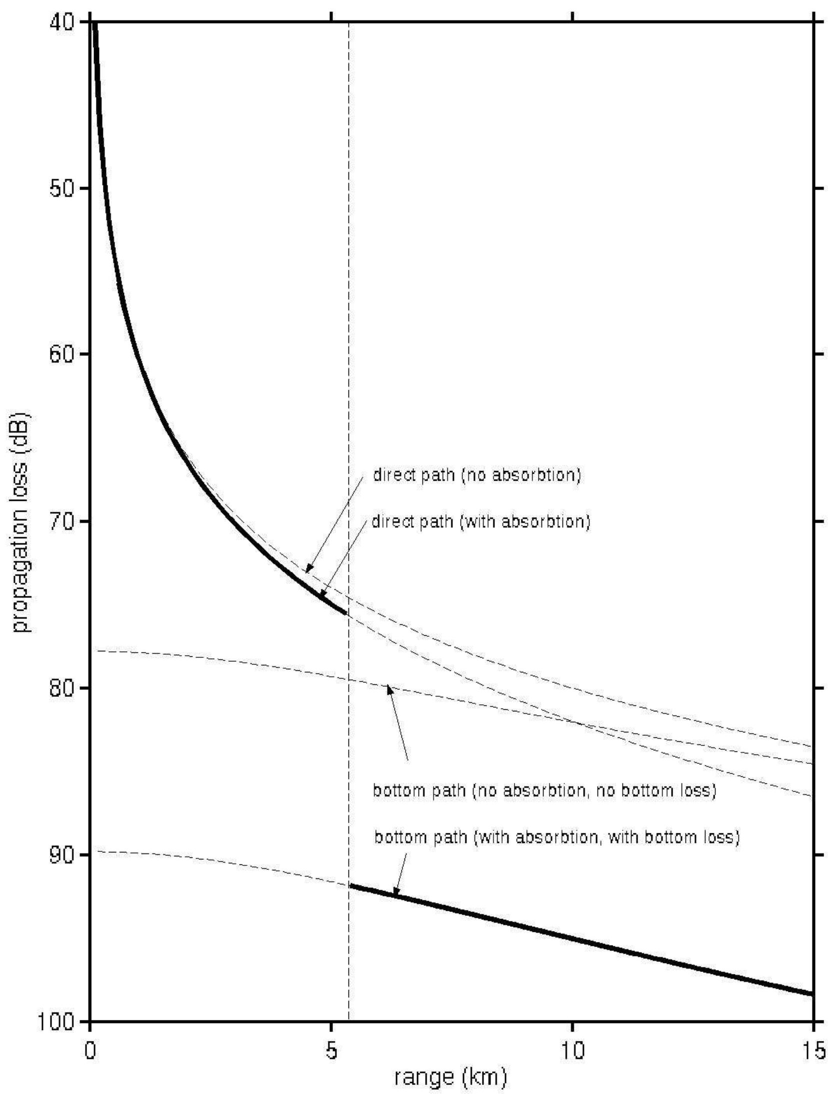

Figure 12

Deep water: sound channel propagation

The measured deep-water sound speed profile given in figure 13 below exhibits a minimum at a depth of approximately 700 m. This is called the 'deep sound channel'.

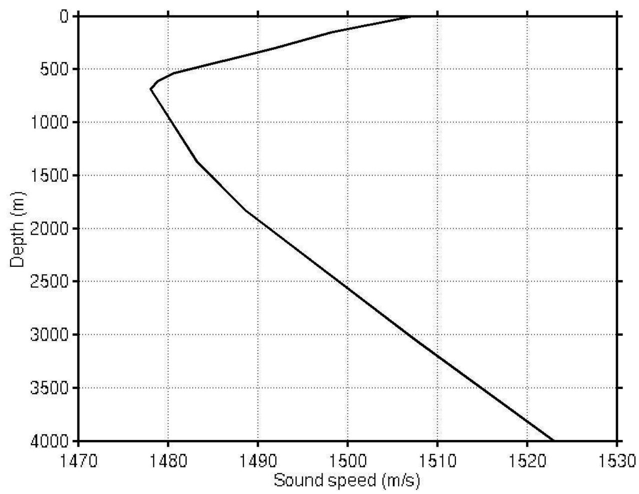

Figure 13

Sound rays launched from a source at this depth oscillate around this depth, thereby focussing the sound as depicted by the corresponding ray diagram given in figure 14.

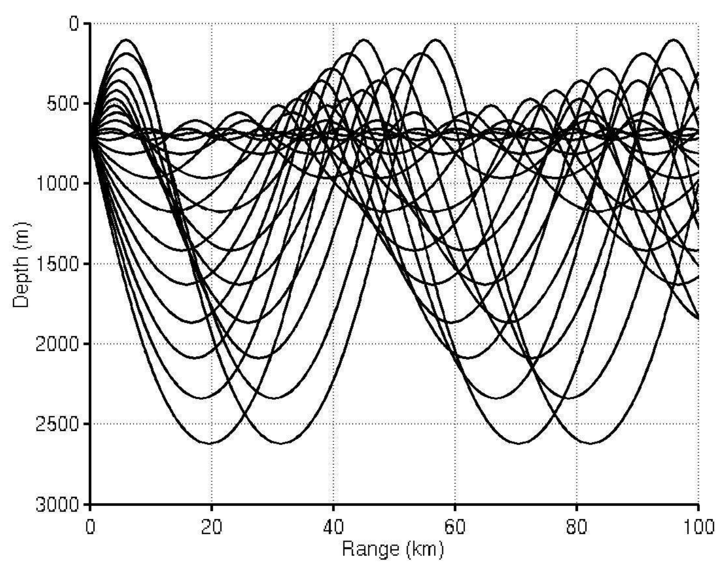

Figure 14

The effect of sound focussing is that after a certain 'transition range' r_0 the geometrical spreading loss changes from spherical into cylindrical. Hence, for $r > r_0$ is PL is given by

$$PL = 10\,^{10}\log r_0 + 10\,^{10}\log r + \alpha r \qquad (24)$$

whereas for $r < r_0$ PL reads

$$PL = 20\,^{10}\log r + \alpha r \qquad (25)$$

Figure 15 shows PL as a function of range for a frequency of 50 Hz (and $z_s = z_r = 700$ m) calculated using a sophisticated computer model (at such a low frequency absorption is negligible). A comparison with the simple equations (24) and (25) gives $r_0 \approx 4$ km (dashed line in the figure). We emphasise that these are very good propagation conditions: doubling the distance only gives a 3 dB extra loss (for ranges $> r_0$).

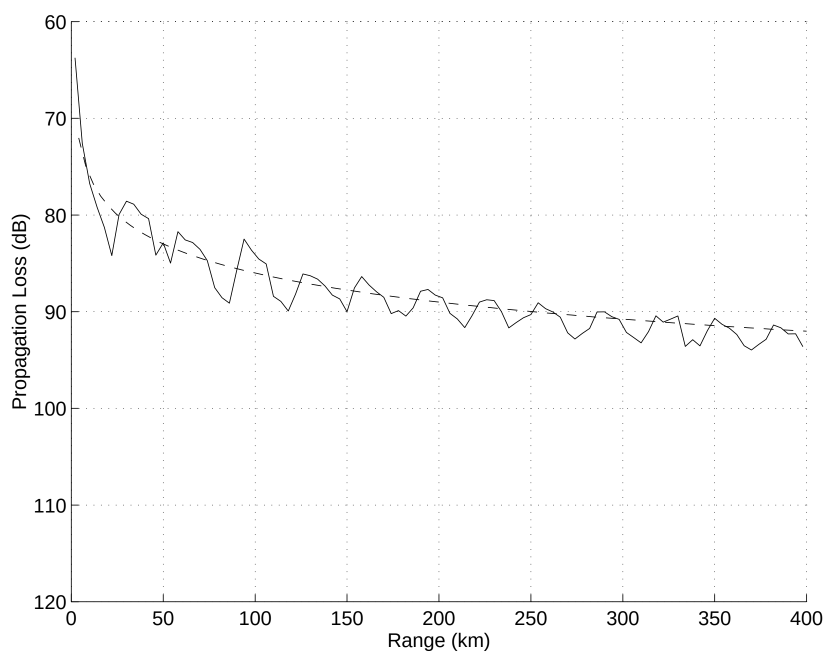

Figure 15

Figure 16 gives PL versus range for a frequency of 1000 Hz. The 50 Hz result is also indicated in the figure for comparison (dashed line), thereby showing the effect of absorption.

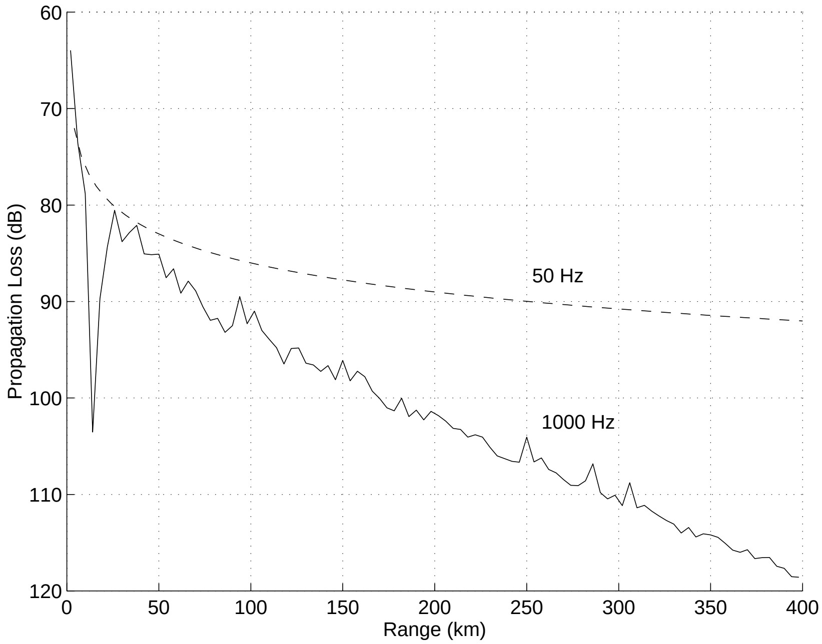
60
70
80
90
100
110
120
Propagation Loss (dB)
0
50
100
150
200
250
300
350
400
Range (km)
50 Hz
1000 Hz

Figure 16