Systeem- en Regeltechniek II

Lecture 2 – System Modeling and Analysis

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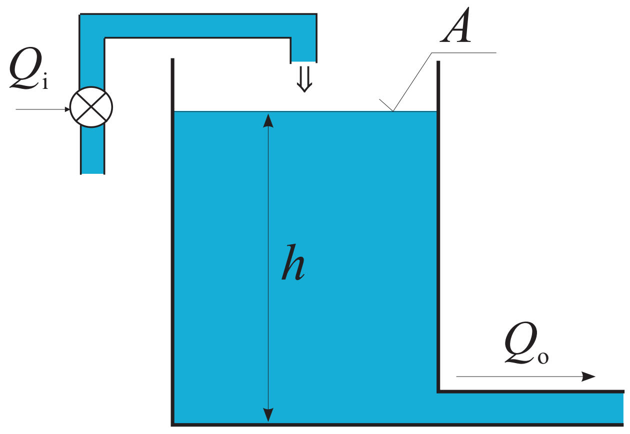
Lecture Outline

Previous lecture: representation of dynamic models as differential equations.

Today:

- Transfer functions.
- State-space models.

Example 3: Liquid Storage Tank



Example 3 (cont'd): Liquid Storage Tank

$$A\dot{h}(t) = Q_i(t) - Q_o(t)$$

$$Q_o(t) = r\sqrt{2gh(t)} = K\sqrt{h(t)}, \quad h \geq 0$$

$$\boxed{A\dot{h}(t) + K\sqrt{h(t)} = Q_i(t)}$$

- Nonlinear differential equation
- Must be linearized for analysis or control design
- Can be used to simulate the process

Laplace Transform – Definition

Transform a signal from time domain to complex domain (s-domain):

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

a number of useful properties

Using Laplace Transform

Differentiation: $f^{(n)}(t) \quad \xrightarrow{\mathcal{L}} \quad s^n F(s)$

Linear differential equation:

$$a_n y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_o y(t) =$$

$$b_m u^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + \cdots + b_o u(t)$$

Linear algebraic equation:

$$\left(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_o\right) Y(s) =$$

$$\left(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_o\right) U(s)$$

Transfer Function

$$\left(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_o\right) Y(s) =$$

$$\left(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_o\right) U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_o}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_o}$$

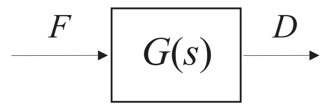
$G(s) = \dfrac{B(s)}{A(s)} \quad \ldots$ transfer function

Example 1 (revisited): Transfer Function

$$m\ddot{d}(t) \; = \; F(t) - b\dot{d}(t)$$

$$ms^2 D(s) \; = \; F(s) - bsD(s)$$

$$G(s) = \frac{D(s)}{F(s)} = \frac{1}{ms^2 + bs} = \frac{1}{s(ms+b)}$$



Example 2 (revisited): Transfer Function

$$L\frac{di(t)}{dt} + Ri(t) = V(t) - K_t\frac{d\theta(t)}{dt} \quad \text{electrical part}$$

$$J\frac{d^2\theta(t)}{dt^2} + b\frac{d\theta(t)}{dt} = K_t i(t) \quad \text{mechanical part}$$

$$(Ls + R)I(s) = V(s) - K_t s\theta(s)$$

$$(Js^2 + bs)\theta(s) = K_t I(s)$$

$$G(s) = \frac{\theta(s)}{V(s)} = \frac{K_t}{s[(Ls+R)(Js+b) + K_t^2]}$$

Example 3 (revisited)

$$A\dot{h}(t) + K\sqrt{h(t)} = Q_i(t)$$

Can we use Laplace transform?

State-Space Models

Introduce state variable $x(t)$ (vector) to parameterize the 'memory' of the system.

- The state contains all information needed to determine future behavior without reference to the derivatives of input and output variables.
- The state is often determined from physical considerations (related to energy storage in the system).
- The dimension n of the state vector is the order of the system.

Linear State-Space Model

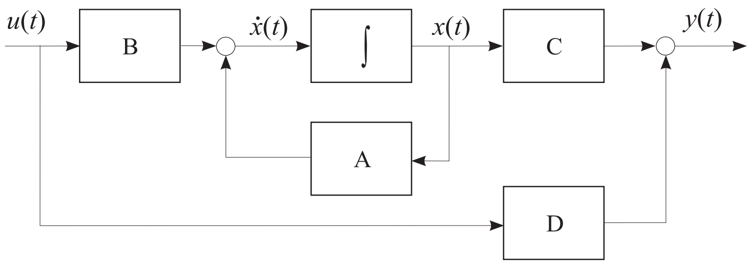
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

A . . . state matrix
B . . . input matrix
C . . . output matrix
D . . . direct transmission matrix

Interpretation:
Derivative of each state is given by a linear combination of states plus a linear combination of inputs. Similarly for the output . . .

State-Space Model: Block Diagram

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$



Example 1 (revisited): State-Space Model

Diff. equation for motion under friction: $m\ddot{d}(t) = F(t) - b\dot{d}(t)$

Introduce velocity: $v(t) = \dot{d}(t)$

Rewrite the above 2nd-order equation as a set of two 1st order DE:

$$\dot{v}(t) = -\frac{b}{m}v(t) + \frac{1}{m}F(t)$$
$$\dot{d}(t) = v(t)$$

Example 1 (revisited): State-Space Model

state: $x(t) = \begin{pmatrix} v(t) \\ d(t) \end{pmatrix}$, input: $u(t) = F(t)$, output: $y(t) = d(t)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{b}{m} & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{m} \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Example 2 (revisited): State-Space Model

$$L\frac{di(t)}{dt} + Ri(t) = V(t) - K_t\frac{d\theta(t)}{dt}$$ electrical part

$$J\frac{d^2\theta(t)}{dt^2} + b\frac{d\theta(t)}{dt} = K_t i(t)$$ mechanical part

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Introduce velocity: $\omega(t) = \dot{\theta}(t)$

Rewrite the above equations as a set of three 1st order DE:

$$\dot{i}(t) = -\frac{R}{L}i(t) - \frac{K_t}{L}\omega(t) + \frac{1}{L}V(t)$$
$$\dot{\omega}(t) = \frac{K_t}{J}i(t) - \frac{b}{J}\omega(t)$$
$$\dot{\theta}(t) = \omega(t)$$

Example 2 (revisited): State-Space Model

state: $x(t) = \begin{pmatrix} i(t) & \omega(t) & \theta(t) \end{pmatrix}^T$, input: $u(t) = V(t)$, output: $y(t) = \theta(t)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{K_t}{L} & 0 \\ \frac{K_t}{J} & -\frac{b}{J} & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Compare to the Input Output Model

$$G(s) = \frac{\theta(s)}{V(s)} = \frac{K_t}{s[(Ls+R)(Js+b)+K_t^2]}$$

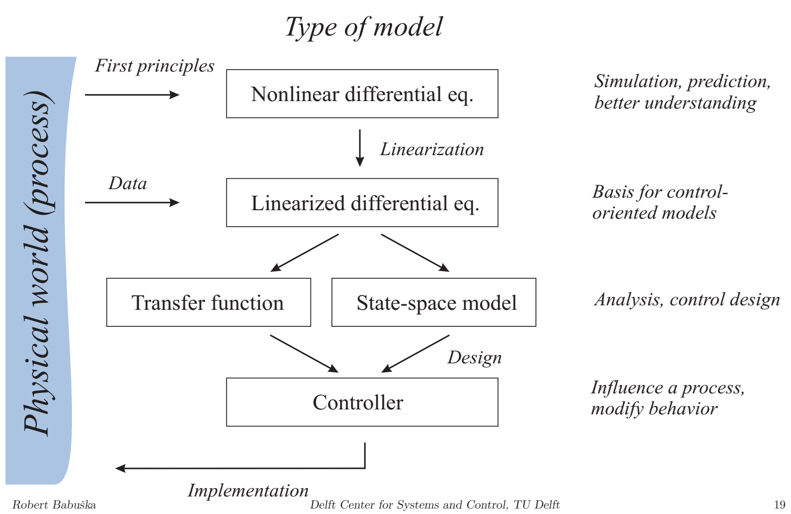
$$\theta(s)\left(LJs^3 + (RJ+Lb)s^2 + (Rb+K_t^2)s\right) = K_t V(s)$$

Corresponds to the following differential equation:

$$LJ\dddot{\theta}(t) + (RJ+Lb)\ddot{\theta}(t) + (Rb+K_t^2)\dot{\theta}(t) = K_t V(t)$$

Note: current i not in the model!
Input-output models do not use internal variables,
instead use higher derivatives of input and outputs.

The big picture



Purpose of Analysis

Analyze the available model in order to:

- Understand the behavior of the process under study.
- Define meaningful specification for the controlled system.
- Give basis for control design choices (controller structure, parameters).

We are mainly interested in:

- Stability of the open-loop process.
- Transient response (impulse, step, ramp).
- Steady-state response (constant or sinusoidal input).