Regeltechniek

Lecture 3 – Stability analysis and transient response

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Lecture Outline

Previous lecture: representation of dynamic models, transfer functions and state-space models.

Today:

- Assignment for the first computer session.
- Stability analysis.
- Transient response.

Information on Computer Sessions

- Compulsory for all students who have not successfully completed it in previous year(s).
- First session in week 4 (calendar week 39)
  - Wednesday 1+2 or 3+4 or Thursday 1+2 or 3+4

  depending on in which group you are - check Blackboard from Monday next week.
- Location: 'Computer room 020' at Civil Engineering.
- Homework preparation required!

Homework for Computer Session

- Read thoroughly the entire handout "Matlab and Simulink for Modeling and Control".
- Work out items a) through e) of Section 5 (by hand).
- If you have never used Matlab before, familiarize yourself with this tool (type 'demo' to start).
- Bring the handout "Matlab and Simulink for Modeling and Control" with you to the computer lab.
- Bring your own laptop with Matlab, if you have one.

Purpose of Analysis

Analyze the available model in order to:

- Understand the behavior of the process under study.
- Define meaningful specification for the controlled system.
- Give basis for control design choices (controller structure, parameters).

We are mainly interested in:

- Stability of the open-loop process.
- Transient response (impulse, step, ramp).
- Steady-state response (constant or sinusoidal input).

Stability – General Notions

- Nonlinear systems – stability of a trajectory (solutions of differential equations).
- Mostly we consider stability of equilibria, i.e., solutions of $0 = f(x_0, u_0)$.
- One system may have many equilibria, some stable, some unstable.
- Linear systems – stability of an equilibrium implies stability of the whole system.

Stability of LTI Systems

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{K \prod_{j=1}^{m}(s - z_j)}{\prod_{i=1}^{n}(s - p_i)}$$

Response to initial conditions: $y(t) = \sum_{i=1}^{n} K_i e^{p_i t}$

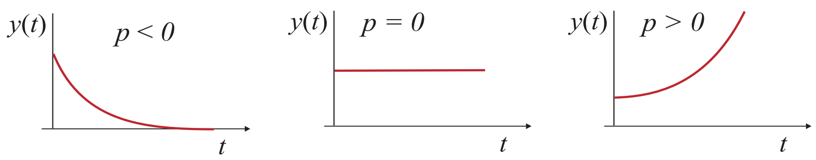
p_i are real or complex poles of the system

The exponential terms decay iff $\mathrm{Re}\{p_i\} < 0$

$=$ the system is stable

Real Poles

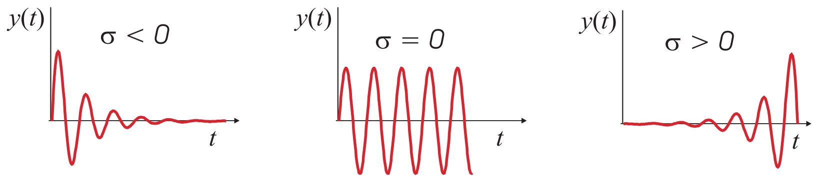
$$y(t) = Ke^{pt}$$



A single pole $p = 0$ . . . system marginally stable
Multiple poles at $p = 0$ . . . system unstable

Complex Poles

$$y(t) = K' e^{\sigma t} \sin(\omega t + \varphi)$$



Transient and Steady-State Response

- Responses to an arbitrary input signal cannot be computed analytically (we have to resort to simulation).
- However, some specific input signals are useful:
  - step response
  - impulse response
  - response to a ramp input (ramp response)
  - response to a sinusoidal input (frequency response)
- Importance both for analysis and for identification of model parameters from measured data.

First-Order Systems: Step Response

$$Y(s) = G(s)U(s) = \frac{1}{\tau s + 1} \cdot \frac{1}{s}$$

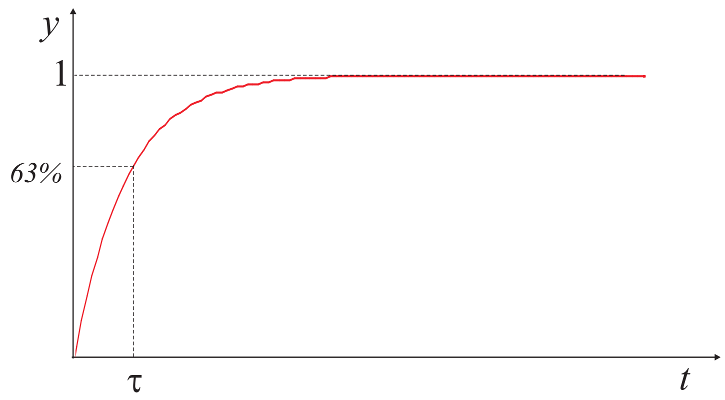
Expand in partial fractions:

$$Y(s) = \frac{1}{s} - \frac{\tau}{\tau s + 1} = \frac{1}{s} - \frac{1}{s + 1/\tau}$$

Corresponding time signal:

$$y(t) = 1 - e^{-t/\tau}$$

First-Order Systems: Step Response



$3\tau \ldots 95\%$, $4\tau \ldots 98\%$, $5\tau \ldots 99\%$ of steady state value

Estimate Parameters From Step Response

- Assume we have a stable first-order process with unknown gain and time constant:

$$G(s) = \frac{K}{\tau s + 1}$$

- Apply a step input to the process (choose a suitable amplitude, not necessarily the unit step).
- Plot the corresponding output and read the parameters from the graph.

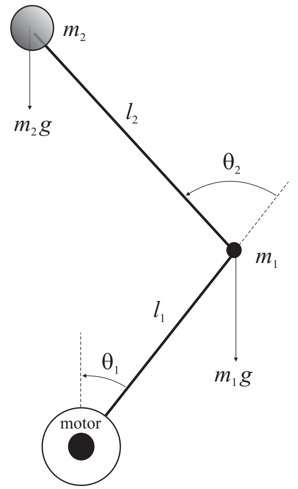
Demo: Rotational Pendulum

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = \begin{bmatrix} k_m u \\ 0 \end{bmatrix}$$

$$M(\theta) = \begin{bmatrix} P_1 + P_2 + 2P_3\cos\theta_2 & P_2 + P_3\cos\theta_2 \\ P_2 + P_3\cos\theta_2 & P_2 \end{bmatrix}$$

$$C(\theta,\dot{\theta}) = \begin{bmatrix} b_1 - P_3\dot{\theta}_2\sin\theta_2 & -P_3(\dot{\theta}_1 + \dot{\theta}_2)\sin\theta_2 \\ P_3\dot{\theta}_1\sin\theta_2 & b_2 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} -g_1\sin\theta_1 - g_2\sin(\theta_1+\theta_2) \\ -g2\sin(\theta_1+\theta_2) \end{bmatrix}$$



Linearized Model

$$G(s) = \frac{\theta_2(s)}{U(s)} = \frac{661.2903(s^2 + 49.05)}{s(s+33.06)(s^2 + 0.6783s + 98.11)}$$

- one pole in origin (pure integration)
- one fast real pole (motor mechanical time constant)
- a pair of poorly damped complex poles
- a pair of complex zeros

First-Order Systems: Ramp Response

$$Y(s) = \frac{1}{\tau s + 1} \cdot \frac{1}{s^2}$$

Expand in partial fractions:

$$Y(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s + 1}$$

Corresponding time signal:

$$y(t) = t - \tau + \tau e^{-t/\tau}$$

First-Order Systems: Impulse Response

$$Y(s) = G(s)U(s) = \frac{1}{\tau s + 1} \cdot 1$$

Expand in partial fractions:

$$Y(s) = \frac{1}{\tau s + 1} = \frac{1/\tau}{s + 1/\tau}$$

Corresponding time signal:

$$y(t) = \frac{1}{\tau} e^{-t/\tau}$$

Relationship Between the Responses

- Ramp response: $G(s) \cdot \frac{1}{s^2}$
- Step response: $G(s) \cdot \frac{1}{s}$ (derivative of ramp response)
- Impulse response: $G(s) \cdot 1$ (derivative of step response)

So far we considered $\tau > 0$ (asymptotically stable first-order system). Work out the impulse, step and ramp response for $\tau < 0$ (unstable system) and for an integrator ($G(s) = 1/s$).

Second-Order System

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k}{(s+\sigma)^2 + \omega_d^2}$$

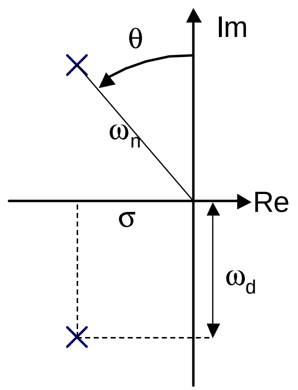
ω_n : undamped natural frequency

ζ : relative damping

σ : attenuation (damping)

ω_d : damped natural frequency

Second-Order System



$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

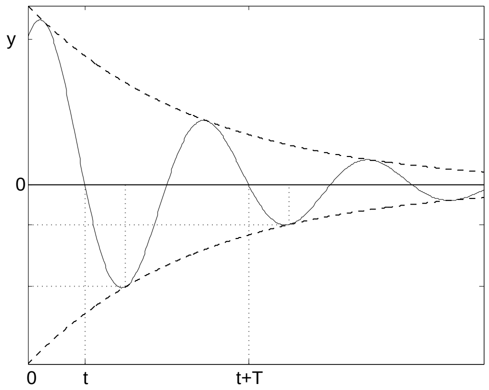
$$\sigma = \zeta\omega_n$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

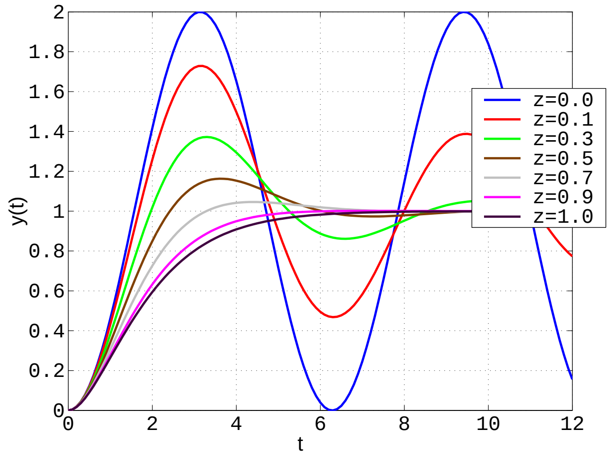
$$\theta = \sin^{-1}\zeta$$

Second-Order System: Time Response

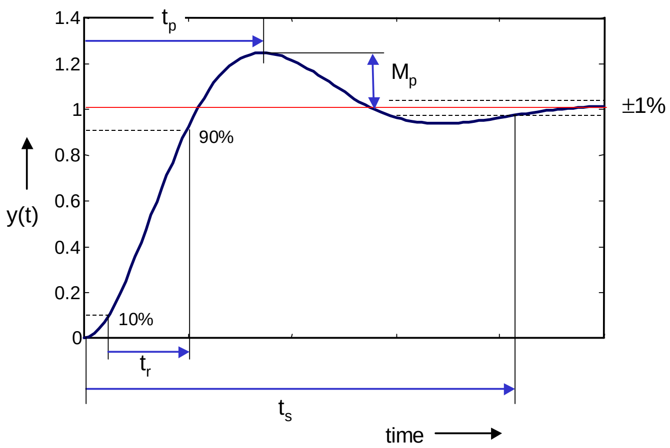
$$y(t) = 1 - e^{-\sigma t}\left(\cos \omega_d t \;+\; \frac{\sigma}{\omega_d}\sin \omega_d t\right)$$



Influence of Damping



Step Response Characteristics



Relation to Natural Frequency and Damping

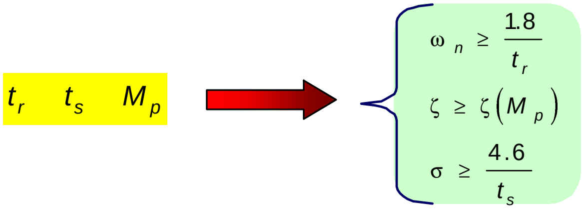
$$t_r = \frac{18}{\omega_n}$$

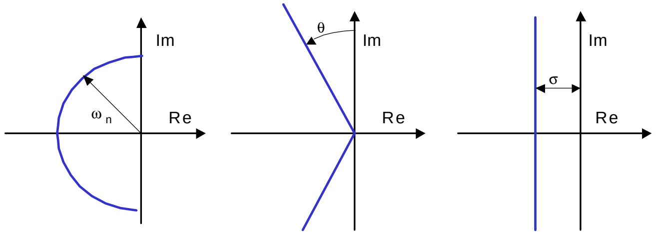
$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$t_s = \frac{4.6}{\zeta\omega_n} \quad \text{for} \quad \pm 1\%$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Performance Specifications for Closed-Loop



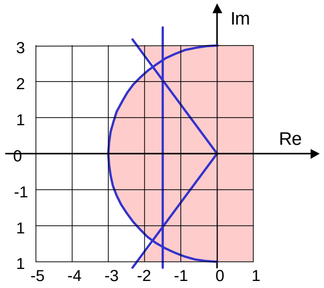


Performance Specifications: Example

$t_r \leq 0.6$ s.
$M_p \leq 10\,\%$
$t_s \leq 3$ s.

$\omega_n \geq 2.8$ rad/s.
$\zeta \geq 0.6$
$\sigma \geq 1.5$

$\omega_n \geq \frac{1.8}{t_r}$
$\zeta \geq \zeta(M_p)$
$\sigma \geq \frac{4.6}{t_s}$



Additional Pole in the System

$$H(s) = \frac{\omega_n^2 \omega_n \gamma}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)\left(s + \gamma\omega_n\right)}$$

The 3$^{\text{rd}}$ order system can be accurately approximated by a second order system if

$$\gamma \geq 10$$

In such a case, the two complex poles are dominant.

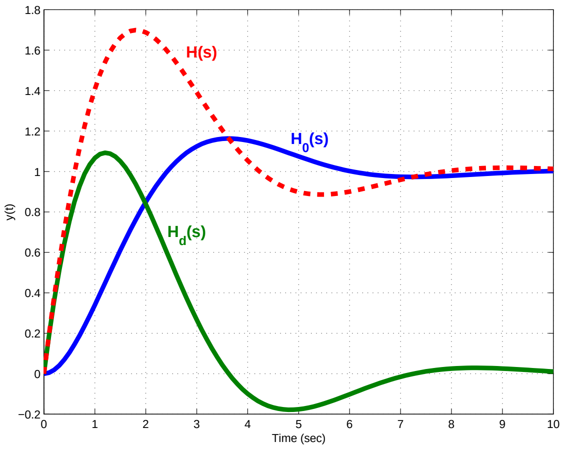
Zero in a Second-Order System

$$H(s) = \frac{\omega_n^2 (bs + 1)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$H(s) = \underbrace{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}_{H_0} + b \cdot \underbrace{\frac{\omega_n^2 s}{s^2 + 2\zeta\omega_n s + \omega_n^2}}_{H_d}$$

Time response of H_d is the derivative of response of H_0.

Step Response With an Extra Zero



Consequences of Extra Zeros

$$H(s) = \underbrace{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}_{H_0} + b \cdot \underbrace{\frac{\omega_n^2 s}{s^2 + 2\zeta\omega_n s + \omega_n^2}}_{H_d}$$

For large $b \rightarrow$ overshoot M_p increases

If $b < 0$ (zero in RHP) initial response is negative

(such systems are called nonminimum-phase systems or inverse-response systems)

Nonminimum-Phase Response