Systeem- en Regeltechniek II

Lecture 5 – The Root Locus Method

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Lecture Outline

Previous lecture: Feedback control, system type, PID control, parameter tuning.

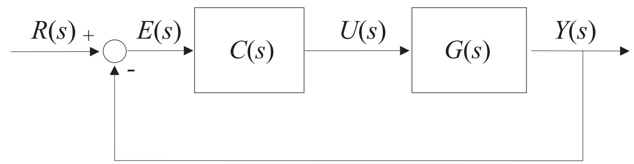
Today:

- Overview of control design methods.
- The root locus (RL) method – basic idea.
- Properties of the RL, sketching for simple systems.
- RL for controller design and closed-loop analysis.
- Matlab and Simulink.

Controller Design Methods: Overview

- Placing the poles in the complex plane
  - Root locus: poles vary as a function of one parameter
  - Pole placement for state feedback (state-space models)
- Shaping the frequency response of closed loop
  - Bode plot: frequency domain design
- Minimizing a cost function of closed loop
  - Linear quadratic control (not in this course)

The Root Locus Method: Setting

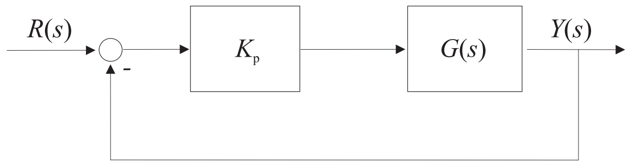


Closed-Loop TF: $\dfrac{Y(s)}{R(s)} = \dfrac{G(s)C(s)}{1 + G(s)C(s)}$

Characteristic equation: $1 + G(s)C(s) = 0 \quad \Rightarrow \quad 1 + L(s) = 0$

Parameterize as: $\boxed{1 + K\bar{L}(s) = 0}$, where $K \in \mathbb{R}$

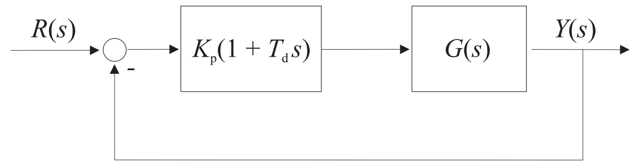
Parameterization: Example 1



$\bar{L}(s) = G(s)$ process

$K = K_p$ proportional gain

Parameterization: Example 2



$\bar{L}(s) = G(s)(1 + T_d s)$ process and normalized PD controller

$K = K_p$ overall gain

etc., PID, lead, lag, . . . (we will come back to these later)

Root Locus: Underlying Question

Given:

$$1 + K\bar{L}(s) = 0$$

How does the location of the closed-loop poles change as a function of K?

In other words: find the set of roots of $(1 + K\bar{L}(s) = 0)$

for varying gain $0 \leq K < \infty$

Useful for both analysis and design!

Root Locus: Solutions

- Analytical: convenient only for first and second-order loop transfers, e.g., $L(s) = \dfrac{1}{s(s+1)}$
- Numerical: compute poles numerically for selected values of K (see the Matlab session)
- Graphical: draw curves along which the closed-loop poles move in the complex plane as K changes (sketch by hand or plot using a computer)

Properties of the Root Locus

(in the sequel we drop the bar from $\bar{L}(s)$, for convenience)

$$1 + KL(s) = 0 \quad \Rightarrow \quad L(s) = -\frac{1}{K}$$

As K is real positive and $L(s)$ complex, the phase:

$$\angle L(s) = 180^\circ + k \cdot 360^\circ \qquad \text{with } k \text{ integer}$$

for all s belonging to the root locus

This is called the phase condition.

Properties of the Root Locus

- The RL has n branches, where n is the number of poles (the order) of $L(s)$.
- For $K = 0$, the branches start in the poles of $L(s)$ and for $K \to \infty$ end in the zeros of $L(s)$:

$$1 + K\,\frac{B(s)}{A(s)} = 0 \quad \Rightarrow \quad A(s) + KB(s) = 0 \quad \Rightarrow \quad \frac{B(s)}{A(s)} = -\frac{1}{K}$$

- If there are fewer zeros than poles (pole excess: $n - m$), the $n - m$ branches go asymptotically to ∞, at angles:

$$\phi_k = \frac{180^\circ + 360^\circ(k-1)}{n-m}, \quad \text{for} \quad k = 1, 2, \ldots, n-m$$

Properties of the Root Locus

- The asymptotes radiate out from point α on the real axis, given by:

$$\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$$

- The RL on the real axis is always left of an odd number of poles and zeros.

Properties of the Root Locus

There are four more properties (rules) for accurately sketching the RL of any system by hand (see page 248 in the book).

In practice, we will plot the RL with Matlab. The rules are then useful to check that we are getting a meaningful plot (typos and other errors do creep in!).

Root Loci of Several Elementary Systems

$$L(s) = \frac{1}{s+a} \quad \text{for} \quad a>0, \quad a=0, \quad a<0$$

$$L(s) = \frac{s+b}{s+a} \quad \text{for} \quad b>0, \quad b=0, \quad b<0$$

$$L(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad \text{for} \quad \text{all} \quad \zeta, \omega$$

$$L(s) = \frac{1}{s^3}, \quad L(s) = \frac{1}{s(\tau s+1)}$$

Know how to sketch RL of first, second and third-order systems with real zeros by hand!

RL for Analysis and Design

1. Qualitative analysis: can a given controller type (P, PI, etc.) stabilize a given system?

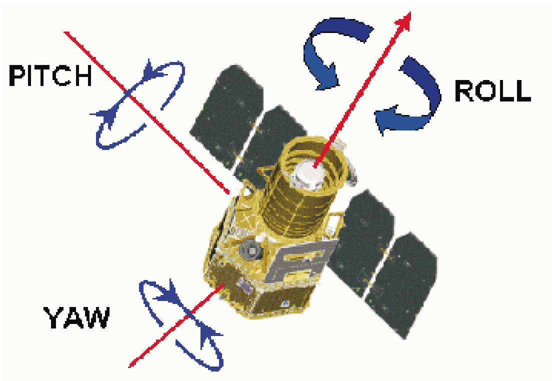
2. Controller's structure: add zero(s) in order to reduce the number of asymptotes and therefore change their angles. Recall the angles:

   $$\phi_k = 180^\circ + 360^\circ(k-1), \quad \text{for} \quad k = 1, 2, \ldots, n-m.$$

   where $n-m$ is the pole-zero excess.

3. Controller's parameter: find K such that desired closed loop specifications are obtained

Example: Satellite Attitude Control



Example: Satellite Attitude Control

$\ddot{\theta}(t) = kT(t)$ where $k > 0$ is a known constant

$\theta(t)$ ... attitude angle (output to be controlled)
$T(t)$ ... thrust (manipulated input)

Transfer function (for $k = 1$): $G(s) = \dfrac{\Theta(s)}{T(s)} = \dfrac{1}{s^2}$

Compare P controller and PD controller.

Use the `rltool` command in Matlab.