# Systeem- en Regeltechniek II

**Lecture 6 – Root Locus, Frequency Response**

**Robert Babuška**

*Delft Center for Systems and Control*
*Faculty of Mechanical Engineering*
*Delft University of Technology*
*The Netherlands*

e-mail: r.babuska@dcsc.tudelft.nl
www.dcsc.tudelft.nl/~babuska
tel: 015-27 85117



# Lecture Outline

Previous lecture: The root locus method, analysis, design.

Today:

- Remarks on the computer session.
- RL: additional examples.
- Realistic PID controller.
- Frequency response.



# Matlab / Simulink Computer Session

- Do your homework:
  - Read the entire handout (know what to do).
  - Work out by hand items a) through e) of Section 5.
- Bring the handout with you to the computer lab.
- Be on time, please, 10 min in advance.

You may bring your own laptop to the lab, if you want.



# RL: Effect of Parameter Change

Consider our DC motor in a position control loop:

1. Use root locus to design a P controller for the nominal system with:

$$K_t = 0.5,\ R = 1,\ L = 0,\ b = 0.1,\ J = 0.01$$

2. Use root locus to analyze how the closed-loop poles change when the moment of inertia $J$ of the load changes.

# DC Motor: Root Locus for P Control

$$G(s) = \frac{\theta(s)}{E(s)} = \frac{K_t}{s[(Ls+R)(Js+b)+K_t^2]}$$

Consider $L = 0$:

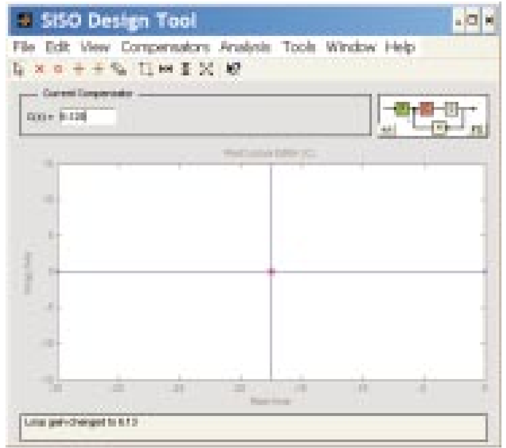
$$G(s) = \frac{\theta(s)}{E(s)} = \frac{K_t}{s[(JRs+bR)+K_t^2]} = \frac{k}{s(s+a)}$$

with $\quad k = \frac{K_t}{JR}, \quad a = \frac{bR+K_t^2}{JR}$

Design a proportional controller such that the closed loop has a double real pole (use `rltool` in Matlab).



# DC Motor: Root Locus for P Control

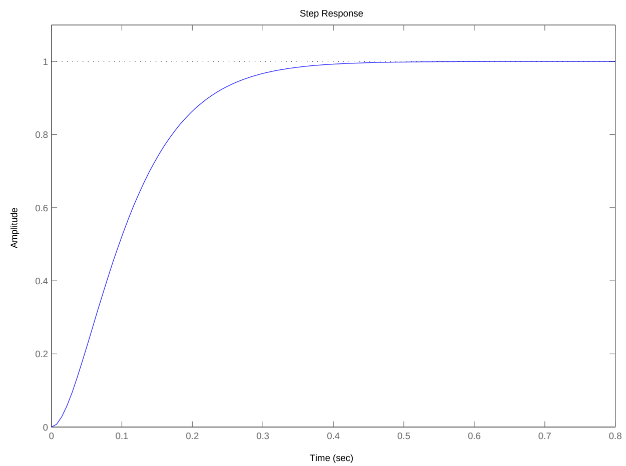



$K_p = 6.125$

$p_{1,2} = -17.5$

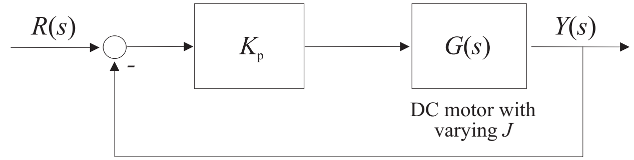


# DC Motor: Closed-Loop Step Response






# RL for Analysis: Varying Moment of Inertia




$L(s) = G(s)K_p$ process and fixed controller in series

$K$ will represent influence of varying moment of inertia $J$

# RL for Analysis: Varying Moment of Inertia

$$G(s)K_p = \frac{k_n}{s(s+a_n)}$$ for the given controller gain $K_p$:

$$k_n = \frac{K_t K_p}{J_n R}, \qquad a_n = \frac{bR + K_t^2}{J_n R} \quad \text{with} \quad J_n = 0.01$$

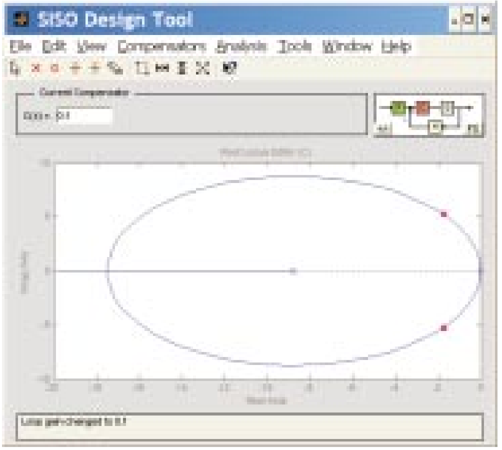
Dividing $J_n$ by factor $K$ means multiplying $a_n$ and $k_n$ by $K$.

Characteristic equation:

$$s^2 + K(sa_n + k_n) = 0 \quad \Rightarrow \quad 1 + K\frac{sa_n + k_n}{s^2} = 0$$



# DC Motor: Root Locus for Varying J



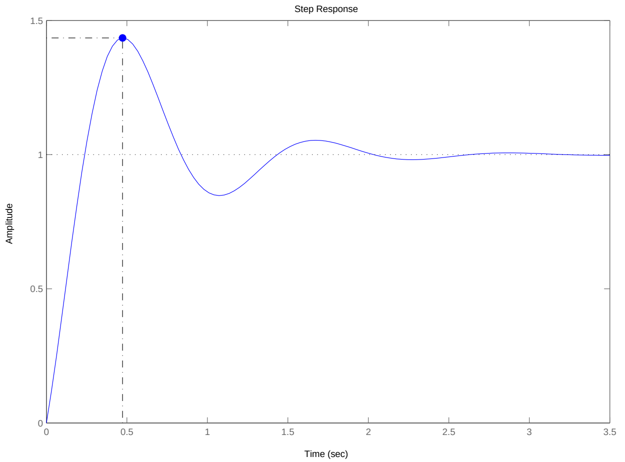

$K = 0.1$

$\Downarrow$

$J = 10 \cdot J_n$



# Large Inertia: Closed-Loop Step Response






# Proper Systems

A system $G(s) = \frac{B(s)}{A(s)}$ for which $\deg A(s) \geq \deg B(s)$

is called proper (has not more zeros than poles).

In reality only proper systems exist!

Consequence:

the 'textbook' form of the PD controller cannot be realized:

$$C(s) = K_p(1 + T_d s)$$

# More Realistic PD Controller

Filtered derivative:

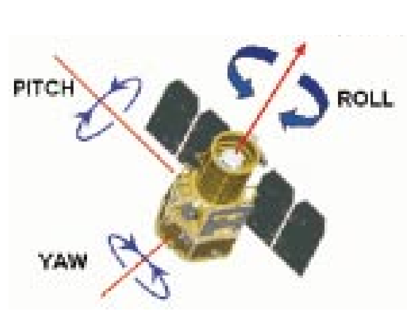
$$C(s) = K_p\left(1 + \frac{T_d s}{(T_d/N)s + 1}\right)$$

where $N$ is typically in the range $10 - 20$.

This means that an additional pole is introduced far left on the real axis.



# Satellite Attitude Control Revisited




Transfer function: $G(s) = \dfrac{\Theta(s)}{T(s)} = \dfrac{1}{s^2}$

Compare the RL for ideal and realistic PD controller.



# PID Controller Used in Practice

$$U(s) = K_p\left(E(s) + \frac{1}{sT_i}E(s) - \frac{T_d s}{(T_d/N)s + 1}Y(s)\right)$$

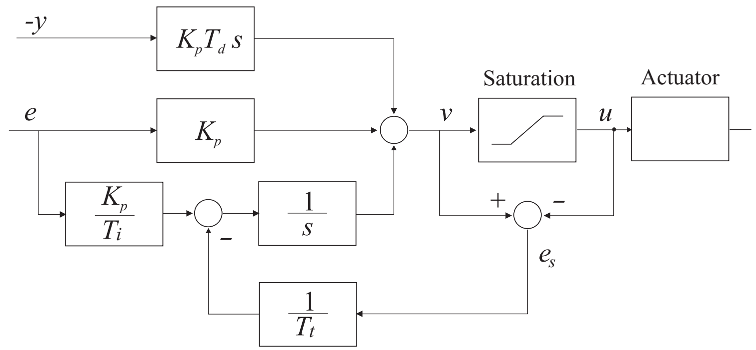
– Derivative action applied to $-Y(s)$ instead of $E(s)$.

– Anti-windup scheme used for the integral action

(prevent integration when the actuator becomes saturated).



# Anti-Windup Tracking Scheme

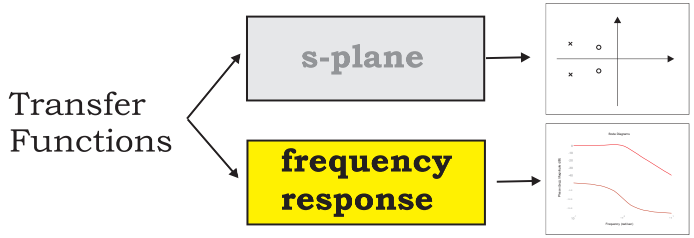


# Root Locus: Homework Assignments

- Read Chapter 5 of the book by Franklin et al.
- Sketch root loci of first, second and third-order systems with real zeros and both real and complex poles by hand.
- For Examples 5.1 through 5.8 in the book verify the results by using Matlab.
- Problems at the end of Chapter 5: work out problems 5.1 and 5.2 by hand and a selection of the remaining problems by using Matlab.



# Representations of Transfer Functions



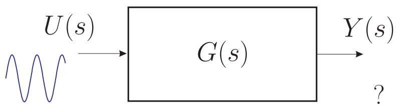

Frequency response:

- Bode plot
- Nyquist plot



# Frequency Response: Setting

Consider a linear time invariant system:




Input: $u(t) = M \sin \omega t$

What is the steady-state output?



# Frequency Response: Laplace Transform

$$u(t) = M \sin \omega t$$

$$U(s) = \mathcal{L}\{u(t)\} = \frac{M\omega}{s^2 + \omega^2} = \frac{M\omega}{(s + j\omega)(s - j\omega)}$$

$$Y(s) = G(s)U(s) = G(s)\frac{M\omega}{(s + j\omega)(s - j\omega)}$$

$$y(t) = ?$$

## Partial Fraction Expansion

General Laplace transform of the output:

$$Y(s) = \sum_{i=1}^{n}\sum_{j=1}^{m_i}\frac{K_{ij}}{(s-p_i)^j} + \frac{K}{s-j\omega} + \frac{K^*}{s+j\omega}$$

Corresponding time signal:

$$y(t) = \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{m_i} K_{ij}t^j e^{p_i t}}_{\text{transient}} + \underbrace{Ke^{j\omega t} + K^* e^{-j\omega t}}_{\text{periodic signal}}$$



## Compute Coefficients K and K*

In steady state:

$$Y(s) = \frac{K}{s-j\omega} + \frac{K^*}{s+j\omega}$$

$$\begin{aligned} K &= Y(s)(s-j\omega)\big|_{s=j\omega} \\ &= G(s)\frac{M\omega}{(s+j\omega)(s-j\omega)}(s-j\omega)\Big|_{s=j\omega} \\ &= G(s)\frac{M\omega}{s+j\omega}\Big|_{s=j\omega} = G(j\omega)\frac{M}{2j} \end{aligned}$$



## Compute Coefficients K and K*

$$K = \frac{M}{2j}G(j\omega) = \frac{M}{2j}\left|G(j\omega)\right| e^{j\angle G(j\omega)}$$

and

$$K^* = -\frac{M}{2j}G(-j\omega) = -\frac{M}{2j}\left|G(j\omega)\right| e^{-j\angle G(j\omega)}$$

$$\begin{aligned} Ke^{j\omega t} + K^* e^{-j\omega t} &= \\ & M\left|G(j\omega)\right| \frac{e^{j(\omega t + \angle G(j\omega))} - e^{-j(\omega t + \angle G(j\omega))}}{2j} \\ &= M\left|G(j\omega)\right| \sin(\omega t + \angle G(j\omega)) \end{aligned}$$



## Frequency Response: Summary

$u(t) = M\sin\omega t$

$y(t) \simeq M\left|G(j\omega)\right|\sin(\omega t + \angle G(j\omega))$

$\left|G(j\omega)\right|$ . . . magnitude (gain)

$\angle G(j\omega)$ . . . phase