Systeem- en Regeltechniek II

Lecture 9 – Lead and Lag Compensators

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Lecture Outline

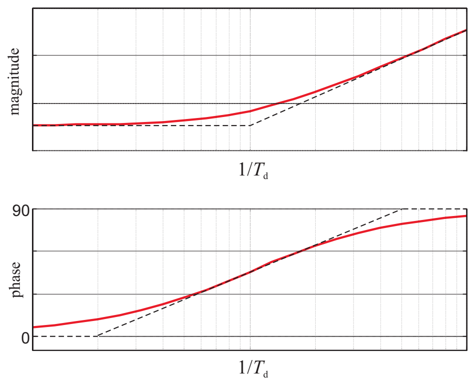
Previous lecture: Bode plots, stability, stability margins.

Today:

- PID controller design.
- Lead and lag compensators.
- Design example - hydraulic actuator.

PD Controller: Bode Plot

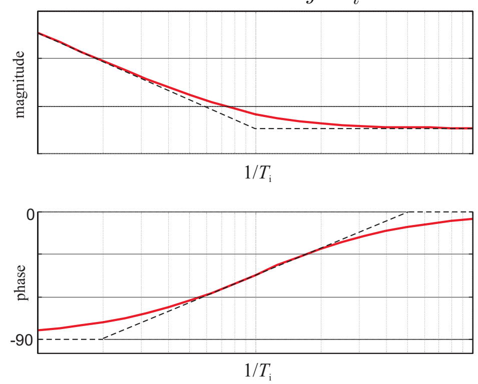
$$C(j\omega) = K_p(1 + j\omega T_d)$$



D-action: adds phase → improves phase margin (damping)!

PI Controller: Bode Plot

$$C(j\omega) = K_p(1 + \frac{1}{j\omega T_i})$$



I-action: adds gain → improves steady-state behavior!

PID Controller

Parallel form (see earlier lectures):

$$C(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

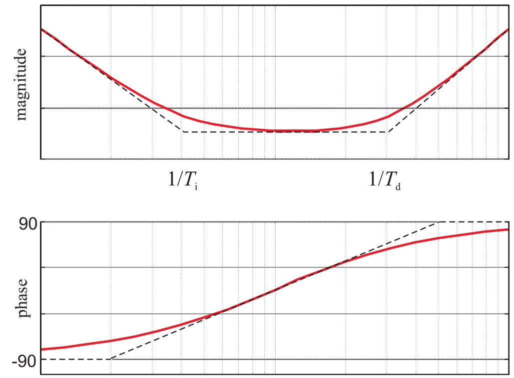
Serial form (more commonly used in FD design):

$$C(s) = K_p\left(1 + \frac{1}{T_i s}\right)(1 + T_d s)$$

$$= \frac{K_p(T_i s + 1)(T_d s + 1)}{T_i s}$$

PID Controller: Bode Plot

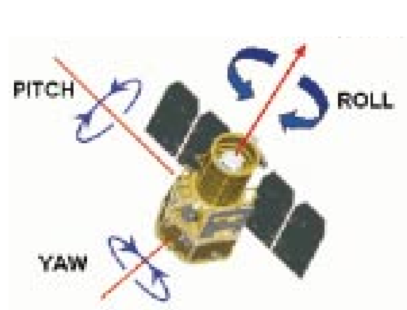
$$C = \frac{K_p(T_i s + 1)(T_d s + 1)}{T_i s}$$



PID Controller Design

- Adjust the proportional gain to get the required crossover frequency and/or steady-state tracking error.
- If needed, use the derivative action to add phase in the neighborhood of ω_c in order to increase the phase margin.
- If needed, use the integral action to increase the gain at low frequencies in order to guarantee the required steady-state tracking error.

Example: PD Satellite Attitude Control

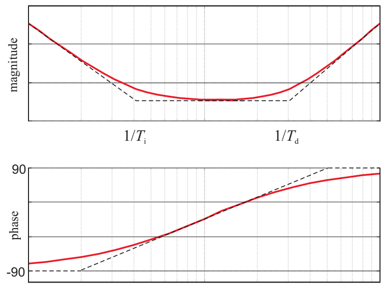


Transfer function: $G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{s^2}$

Performance specs: bandwidth of ≈ 0.2 rad/s, good damping.

Drawbacks of the PID Controller

- The derivative action introduces very large gain for high frequencies (noise amplification).
- The integral action introduces infinite gain for zero frequency (it is open-loop unstable) if the loop is broken.



Lead and Lag Compensation

Lead compensator:

$$C_{\text{lead}}(s) = \frac{T_{\text{lead}}s + 1}{\alpha T_{\text{lead}}s + 1} \qquad \alpha < 1$$

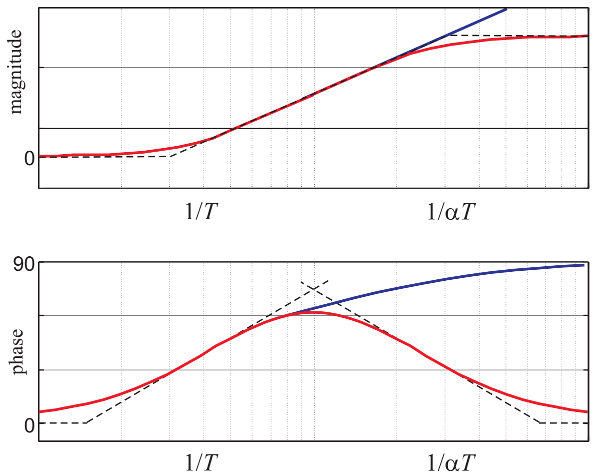
Lag compensator:

$$C_{\text{lag}}(s) = \beta \frac{T_{\text{lag}}s + 1}{\beta T_{\text{lag}}s + 1} \qquad \beta > 1$$

Lead-lag compensator:

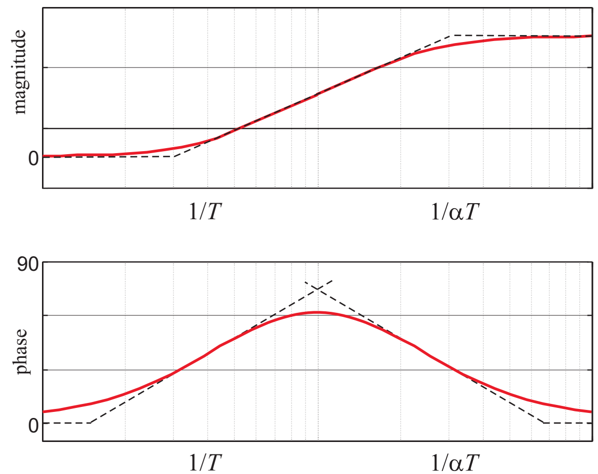
$$C(s) = \beta \frac{T_{(\text{lead}}s + 1)(T_{\text{lag}}s + 1)}{(\alpha T_{\text{lead}}s + 1)(\beta T_{\text{lag}}s + 1)}$$

Lead Compensator: Bode Plot



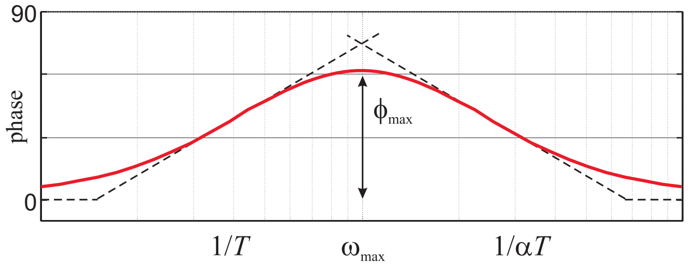
$$\frac{Ts + 1}{\alpha Ts + 1}$$

Lead Compensator vs. PD Controller



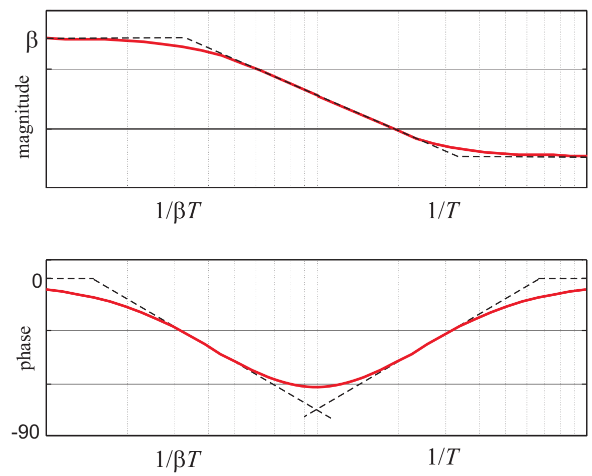
$$\frac{Ts + 1}{\alpha Ts + 1}$$

Lead Compensator – Maximal Phase Lead



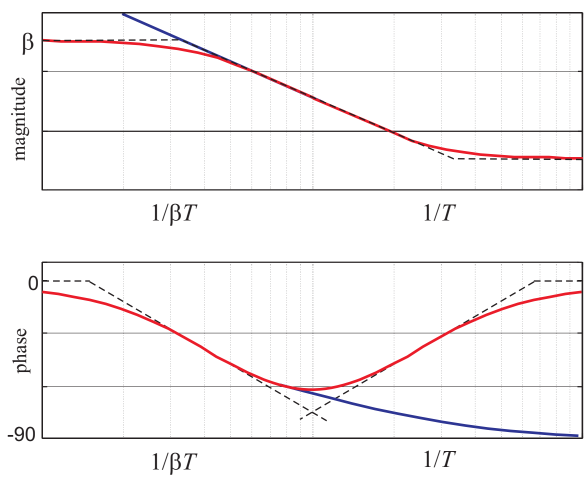
$$\omega_{\max} = \frac{1}{T\sqrt{\alpha}} \qquad \sin\phi_{\max} = \frac{1-\alpha}{1+\alpha} \quad \Rightarrow \quad \alpha = \frac{1-\sin\phi_{\max}}{1+\sin\phi_{\max}}$$

Lag Compensator: Bode Plot



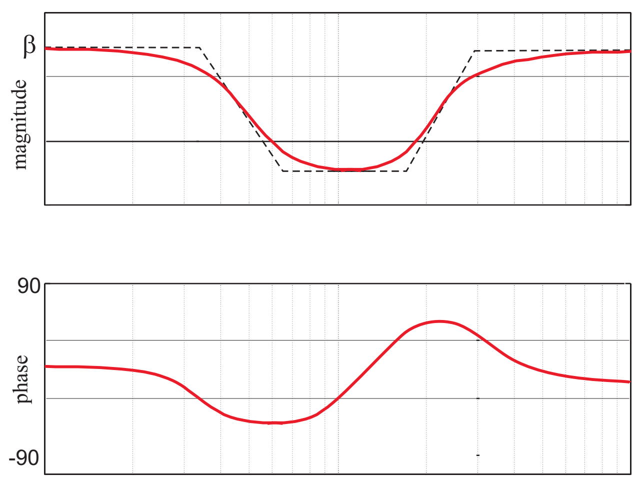
$$\beta\frac{Ts+1}{\beta Ts+1}$$

Lag Compensator vs. PI Controller



$$\beta\frac{Ts+1}{\beta Ts+1}$$

Lead-Lag Compensator: Bode Plot



Closed Loop Control

Closed loop TF:

$$G_{cl}(s) = \frac{G(s)D(s)K}{1 + G(s)D(s)K}$$

$D(s)$ is either the lead, the lag or the lead-lag compensator

– lead compensator = realistic PD controller

– lag compensator = gain-limited PI controller

Lead Compensator Design

1. Determine the crossover frequency. Typically:

$$\omega_c \leq \omega_{bw} \leq 2\omega_c$$

2. Calculate how much extra phase must be added by the lead compensator at the crossover frequency. Compute:

$$\alpha = \frac{1 - \sin\phi_{\max}}{1 + \sin\phi_{\max}} \qquad \frac{1}{T_{\text{lead}}} = \omega_c\sqrt{\alpha}$$

3. Compute the overall controller gain K such that the required ω_c is obtained.

4. Check whether the specs are met, if not, revise choices.

Lag Compensator Design

1. Determine the crossover frequency. Typically:

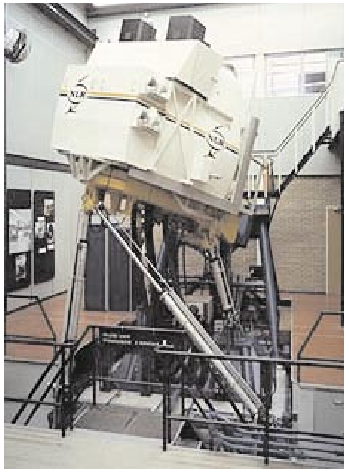
$$\omega_c \leq \omega_{bw} \leq 2\omega_c$$

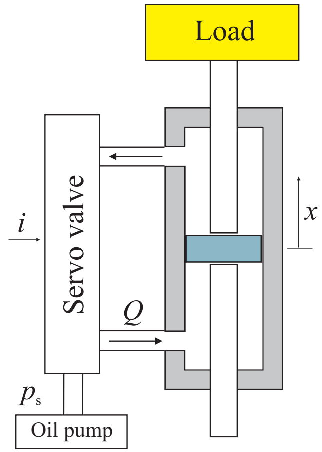
2. Determine β to meet the steady-state requirements.

3. Choose $T_{\text{lag}} \in \left[\frac{1}{0.5\omega_c}, \frac{1}{0.1\omega_c}\right]$.

4. Check whether the specs are met, if not, revise choices, iterate on the design.

Design Example: Hydraulic Actuator





Hydraulic Actuator – Physical Model

$$M\ddot{x} + b\dot{x} + Mg = A_p p$$

$$\frac{V}{E_o}\dot{p} + L_e p + A_p\dot{x} = Q$$

$$Q + \tau\dot{Q} = \left(K_v\sqrt{1 - \frac{|p|}{p_s}}\right) i$$

x – piston position (to be controlled)

p – oil pressure in the cylinder

Q – oil flow-rate

i – servo valve current (control input)

Hydraulic Actuator – Control Specs

closed-loop bandwidth: $\omega_{bw} \approx 40\,\mathrm{rad/s}$

phase margin: $\mathrm{PM} \approx 60°$

steady-state ramp tracking error: $e_{ss} \leq 0.01\,\mathrm{m/s}$

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rise time: $t_r = 1.8/\omega_{bw} \approx 0.045\,\mathrm{s}$

relative damping: $\zeta \approx \mathrm{PM}/100 \approx 0.6$

overshoot: $M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \approx 10\,\%$

crossover frequency: $\omega_c = \omega_{bw}/2 \approx 20\,\mathrm{rad/s}$

Hydraulic Actuator – Linearized Model

$$G(s) = \frac{X(s)}{I(s)} = \frac{5574416}{s(s+25)(s^2 + 91.53s + 8068)}$$

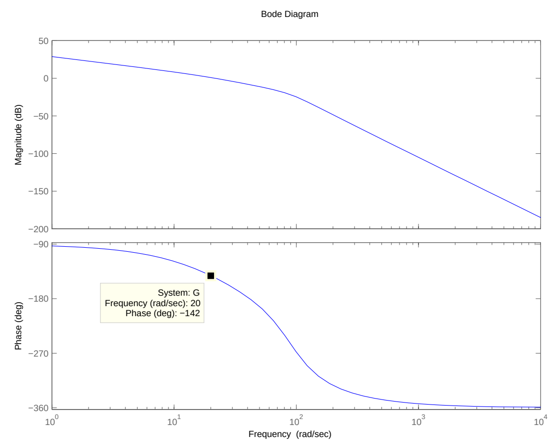
System type: 1 (one pure integrator)

$$K_v = 5574416/(25 \cdot 8068) = 27.58$$

Steady-state error for ramp: $1/K_v = 0.036$

Required steady-state error: 0.01

Bode Plot of Loop TF



Phase(ω_c)= 180-142 = 38 deg → additional 22 deg needed.

Lead Compensator Design

Take additional phase of 27 deg (extra margin of 5 deg):

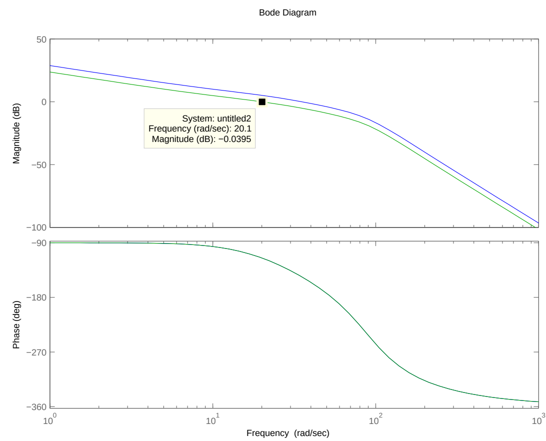
$$\alpha = \frac{1 - \sin(27\pi/180)}{1 + \sin(27\pi/180)} = 0.375$$

$$\frac{1}{T_{\text{lead}}} = \omega_c\sqrt{\alpha} \quad \rightarrow \quad T_{\text{lead}} = 0.08\,s$$

$$C_{\text{lead}} = \frac{T_{\text{lead}}s + 1}{\alpha T_{\text{lead}}s + 1} = \frac{0.08s + 1}{0.03s + 1}$$

$$K = 1/\left|G(j\omega_c)C_{\text{lead}}(j\omega_c)\right| = 0.553$$

Lead-Compensated Loop TF



K = 0.553 (-5.11 dB).

Lag Compensator Design

The lead compensator satisfies the bandwidth and PM specs. However, it cannot meet the steady-state error requirement $e_{ss} = 0.01$:

$$G(s)C_{\text{lead}}(s)K = \frac{5574416}{s(s+25)(s^2+91.53s+8068)} \cdot \frac{0.08s+1}{0.03s+1} \cdot 0.553$$

$$K_v = 5574416 \cdot 0.553/(25 \cdot 8068) = 15.32$$

$$e_{ss} = \frac{1}{K_v} = 0.0653$$

Lag Compensator Design

$$C_{\text{lag}}(s) = \beta\frac{T_{\text{lag}}s + 1}{\beta T_{\text{lag}}s + 1} \qquad \beta > 1$$

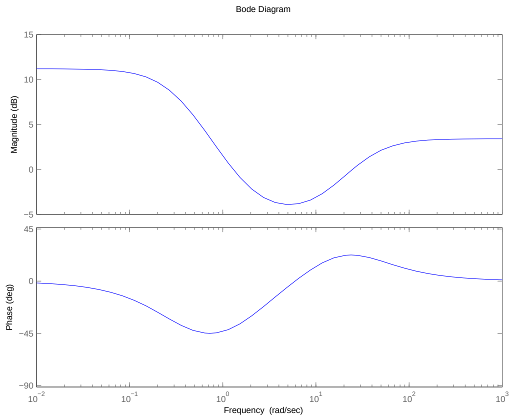
Additional steady-state gain $\beta = 0.0653/0.01 = 6.53$.

Choose $T_{\text{lag}} = 1/(0.1\omega_c) = 0.5\,\text{s}$ (rule of thumb)

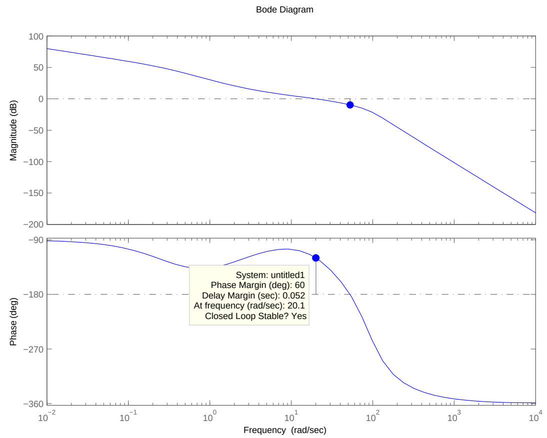
$$C_{\text{lag}}(s) = 6.53 \cdot \frac{0.5s + 1}{3.27s + 1}$$

Lead-Lag Compensator Bode Plot

$$C(s) = K \cdot C_{\text{lag}}(s) \cdot C_{\text{lead}}(s) = 0.553 \cdot 6.53 \cdot \frac{0.5s+1}{3.27s+1} \cdot \frac{0.08s+1}{0.03s+1}$$

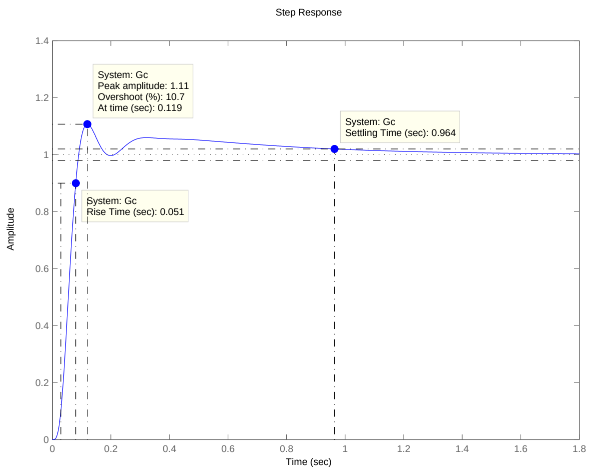


Lead-Lag-Compensated Loop TF



Requirements met!

Hydraulic Actuator – Step Response



Bode Plots: Homework Assignments

- Read Sections 6.1 through 6.7, except for the Nyquist criterion.
- Work out examples in these sections and verify the results by using Matlab.
- Reproduce the derivation of the frequency response as given on the overhead sheets.
- Work out a selection of problems 6.3 – 6.9, and problems 6.16, 6.17, 6.42 – 6.45, 6.55 – 6.57 and verify your results by using Matlab.