Regeltechniek

Lecture 11 – State-space models and state feedback control

Robert Babuška

Delft Center for Systems and Control
Faculty of Mechanical Engineering
Delft University of Technology
The Netherlands

e-mail: r.babuska@dcsc.tudelft.nl
www.dcsc.tudelft.nl/˜babuska
tel: 015-27 85117

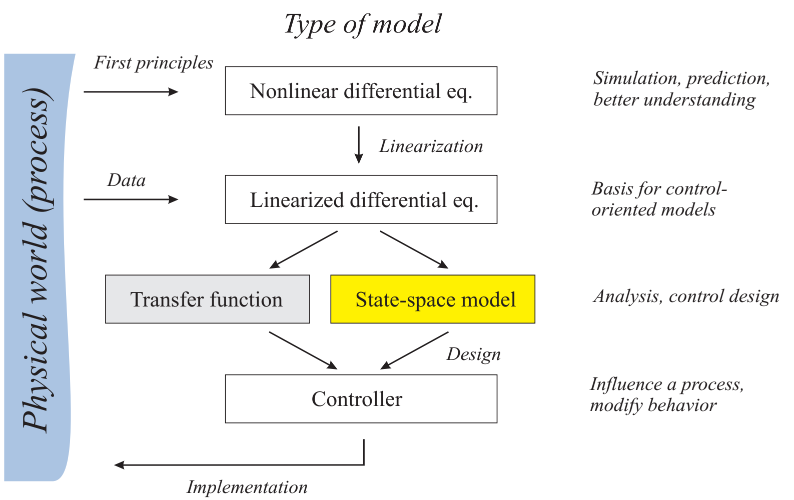
Lecture Outline

Previous lecture: Nyquist plot and stability criterion.
(We are finished with frequency domain methods.)

Today:

- State-space models, representation.
- State feedback, pole placement.

The big picture



Transfer Function vs. State Space Methods

Transfer function	State-space
graphical tools	computational
provide insight	numerical, less insight
interactive / iterative	“numbers in – numbers out”
SISO systems	SISO and MIMO systems

Linear State-Space Model

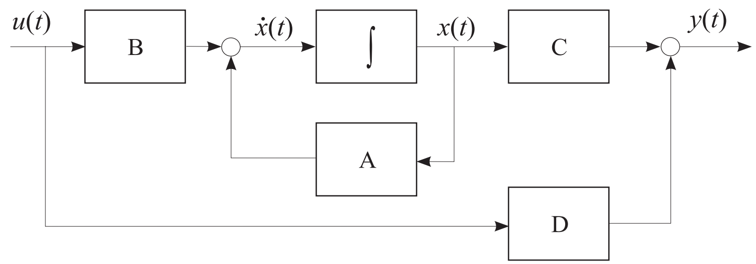
$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

A . . . state matrix
B . . . input matrix
C . . . output matrix
D . . . direct transmission matrix

Interpretation:
Derivative of each state is given by a linear combination of states plus a linear combination of inputs. Similarly for the output . . .

State-Space Model: Block Diagram

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$



DC Motor: State-Space Model

$$L\frac{di(t)}{dt} + Ri(t) = V(t) - K_t\frac{d\theta(t)}{dt} \qquad \text{electrical part}$$

$$J\frac{d^2\theta(t)}{dt^2} + b\frac{d\theta(t)}{dt} = K_t i(t) \qquad \text{mechanical part}$$

Introduce velocity: $\omega(t) = \dot{\theta}(t)$

Rewrite the above equations as a set of three 1st order DE:

$$\begin{aligned}\dot{i}(t) &= -\frac{R}{L}i(t) - \frac{K_t}{L}\omega(t) + \frac{1}{L}V(t)\\ \dot{\omega}(t) &= \frac{K_t}{J}i(t) - \frac{b}{J}\omega(t)\\ \dot{\theta}(t) &= \omega(t)\end{aligned}$$

DC Motor: State-Space Model

state: $x(t) = \begin{pmatrix} i(t) & \omega(t) & \theta(t)\end{pmatrix}^T$, input: $u(t) = V(t)$, output: $y(t) = \theta(t)$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{K_t}{L} & 0\\ \frac{K_t}{J} & -\frac{b}{J} & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + \begin{pmatrix}\frac{1}{L}\\ 0\\ 0\end{pmatrix}u$$

$$y = \begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

Compare to the Input Output Model

$$G(s) = \frac{\theta(s)}{V(s)} = \frac{K_t}{s[(Ls+R)(Js+b)+K_t^2]}$$

$$\theta(s)\left(LJs^3 + (RJ+Lb)s^2 + (Rb+K_t^2)s\right) = K_t V(s)$$

Corresponds to the following differential equation:

$$LJ\dddot{\theta}(t) + (RJ+Lb)\ddot{\theta}(t) + (Rb+K_t^2)\dot{\theta}(t) = K_t V(t)$$

Input-output models do not use internal variables,
instead use higher derivatives of input and output.

State-Space Models → Transfer Function

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Use Laplace:

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

Express $X(s)$:

$$(sI-A)X(s) = BU(s)$$
$$X(s) = (sI-A)^{-1}BU(s)$$

$$Y(s) = CX(s) + DU(s) = \left(C(sI-A)^{-1}B + D\right)U(s)$$

Transfer function: $G(s) = C(sI-A)^{-1}B + D$

Poles of State-Space Models

Transfer function: $G(s) = C(sI-A)^{-1}B + D$

$$(sI-A)^{-1} = \frac{\mathrm{adj}(sI-A)}{\det(sI-A)}$$

Poles = roots of the characteristic equation of A:

$$\det(sI-A) = 0$$

Therefore the poles are the eigenvalues of state matrix A.

State Feedback Control – Main Idea

- Assume a state-space model with all states measured:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = x(t)$$

- Controller = linear combination of states:

$$u(t) = -Kx(t) = -k_1x_1(t) - k_2x_2(t) \; \cdots - k_nx_n(t)$$

- Goal: obtain desired dynamics (e.g., fast, well damped)
- Design parameters: location of closed-loop poles

State Feedback Control – Remarks

- Note that the controller is a static system:

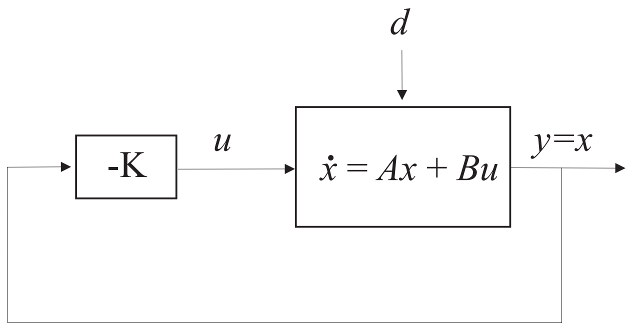
$$u(t) = -Kx(t) = -k_1x_1(t) - k_2x_2(t) \ \cdots - k_nx_n(t)$$

  (as opposed to e.g. a lead-lag compensator or PID).

- Any desired location of closed-loop poles can be obtained (which is not the case with e.g. root-locus)

- Often not all the states are measured (need an observer).

State Feedback Control Scheme



State Feedback Control – Problem

Given the state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

We want to design the state-feedback gain K (a vector)

$$u(t) = -Kx(t)$$

Central question:

How can we compute K, such that the closed-loop poles are at a desired location?

One Possible Solution

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad u(t) = -Kx(t)$$

Construct the closed-loop system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BKx(t) \\ \dot{x}(t) &= \underbrace{(A - BK)}_{A_{cl}} x(t) \end{aligned}$$

Compute the closed-loop characteristic polynomial:

$$\det(sI - A_{cl}) = \det(sI - A + BK) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

Solution: Finding K

Closed-loop characteristic polynomial:

$$\det(sI - A + BK) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

The coefficients a_i are linear functions of the gains k_i.

Define desired characteristic polynomial

(i.e., desired dynamics in terms of poles):

$$d(s) = s^n + d_1 s^{n-1} + \cdots + d_{n-1}s + d_n$$

Compute K by equating coefficients a_i and d_i

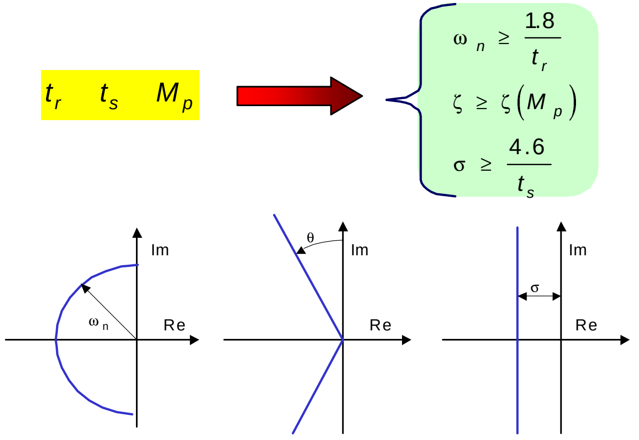
Desired Dynamics (Closed-Loop Poles)

Control goals are typically stated in terms of:

- desired frequency ω_n and damping ζ
  for dominant second-order dynamics
- time domain characteristics:
  rise time, settling time, overshoot

$$G_d(s) = \underbrace{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}_{\text{dominant dynamics}} \cdot \underbrace{\frac{\alpha * \omega_n}{s + \alpha * \omega_n}}_{\text{faster dynamics}} \cdots$$

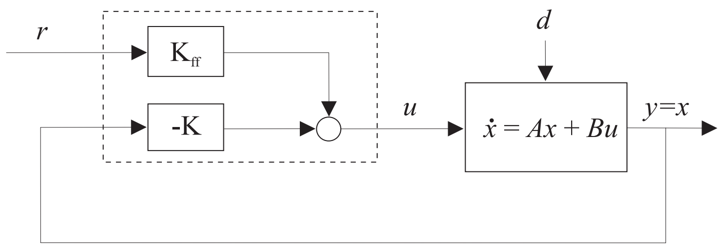
Performance Specifications for Closed-Loop



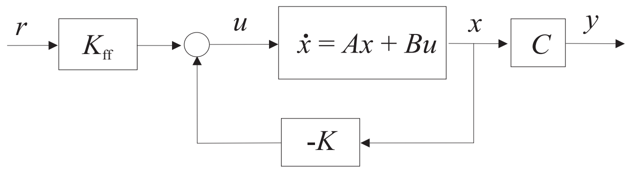
Tracking a Reference Input (Servo)

Goal: respond to a reference signal in a specified way.

Replace $u(t) = -Kx(t)$ by: $u(t) = -Kx(t) + K_{ff}\, r(t)$



Computing the Feedforward Gain



Transfer function from r to y:

$$H_{cl}(s) = C(sI - A + BK)^{-1}BK_{ff} = K_{ff}\frac{b(s)}{d(s)}$$

with: $b(s)$ the process open-loop TF numerator
$d(s)$ is the desired closed loop TF denominator

Computing the Feedforward Gain

$$H_{cl}(s) = C(sI - A + BK)^{-1}BK_{ff} = K_{ff}\frac{b(s)}{d(s)}$$

Given a desired DC gain of the closed loop:

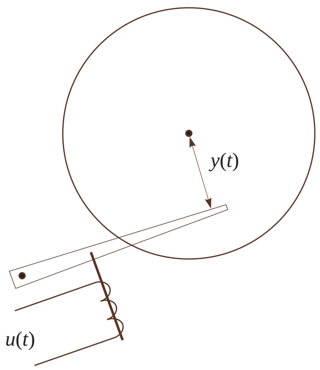
$$H_{cl}(0) = \mathrm{DC}_{\mathrm{des}}$$

Compute:

$$K_{ff} = \frac{d(0)}{b(0)} \cdot \mathrm{DC}_{\mathrm{des}}$$

Example: Disk Drive Arm Control

$$\ddot{y}(t) = \frac{c}{J}u(t)$$



Specifications: settling time < 10 ms, overshoot $< 2\%$

Example: Disk Drive Arm Control

1. Form a state-space model A, B, C, D
2. Analyze open-loop dynamics (poles)
3. Define desired closed loop dynamics $d(s)$
4. Compute state-feedback gain vector K
5. Compute feedforward gain vector K_{ff}