Systeem- en Regeltechniek II

Lecture 12 – State-space models and state feedback control

Robert Babuška

Delft Center for Systems and Control
Faculty of Mechanical Engineering
Delft University of Technology
The Netherlands

e-mail: r.babuska@tudelft.nl
www.dcsc.tudelft.nl/˜babuska
tel: 015-27 85117

Lecture Outline

Previous lecture: State-space models, representation, pole placement.

Today:

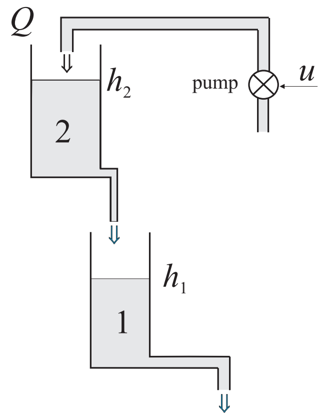
- Coordinate transformation, control canonical form.
- Pole placement, Ackermann's formula.
- DC motor demo.

Is the State-Space Representation Unique?

- For a given system, there is a unique transfer function representing that system (i.e., unique polynomials $b(s)$, $a(s)$).
- Does the same hold for the state-space representation (i.e., for the matrices A, B, C, D)?

Let's have a look at an example . . .

Cascaded Tanks System



Linearized differential equations:

$$\dot{h}_1(t) + 0.5h_1(t) = 0.5h_2(t)$$
$$\dot{h}_2(t) + 0.2h_2(t) = 2u(t)$$

Cascaded Tanks – State-Space Model

$$\dot{h}_1(t) + 0.5h_1(t) = 0.5h_2(t)$$
$$\dot{h}_2(t) + 0.2h_2(t) = 2u(t)$$

State-space model:

$$\dot{x}(t) = \begin{pmatrix} -0.5 & 0.5 \\ 0 & -0.2 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

Cascaded Tanks – Transfer Function

$$\dot{h}_1(t) + 0.5h_1(t) = 0.5h_2(t)$$
$$\dot{h}_2(t) + 0.2h_2(t) = 2u(t)$$

Transfer function:

$$G(s) = \frac{H_1(s)}{U(s)} = \frac{0.5}{s+0.5} \cdot \frac{2}{s+0.2} = \frac{1}{s^2 + 0.7s + 0.1}$$

Corresponding differential equation:

$$\ddot{h}_1(t) + 0.7\dot{h}_1(t) + 0.1h_1(t) = u(t)$$

Cascaded Tanks – State-Space Model (II)

$$\ddot{h}_1(t) + 0.7\dot{h}_1(t) + 0.1h_1(t) = u(t)$$

State-space model:

$$\dot{z}(t) = \begin{pmatrix} -0.7 & -0.1 \\ 1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} z(t)$$

Comparison of State-Space Models

State-space model I (physical):

$$\dot{x}(t) = \begin{pmatrix} -0.5 & 0.5 \\ 0 & -0.2 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u(t) \quad , \quad y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

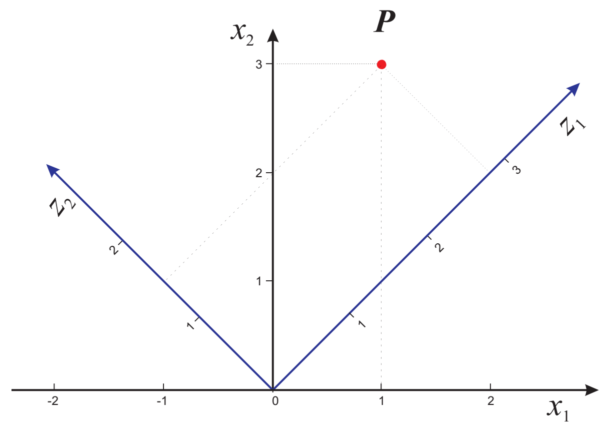
State-space model II (from TF):

$$\dot{z}(t) = \begin{pmatrix} -0.7 & -0.1 \\ 1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \quad , \quad y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} z(t)$$

Uniqueness

- A state-space representation (form) is not unique.
- There are actually infinitely many possible forms.
- How can we transform one into another?
- Are some forms more useful than others?

Coordinate Transformation



$P_x = TP_z$ (P in x-coord. = transf. matrix · P in z-coord.)

Coordinate Transformation

Introduce new state vector z such that:

$$x = Tz$$

where T is a non-singular transformation matrix.

$$\dot{x}(t) = ATz(t) + Bu(t)$$

Transformed model:

$$\dot{z}(t) = \underbrace{T^{-1}AT}_{\tilde{A}} z(t) + \underbrace{T^{-1}B}_{\tilde{B}} u(t)$$

and

$$y(t) = \underbrace{CT}_{\tilde{C}} z(t) + Du(t)$$

Implications of Coordinate Transformation

- Matrices A, B and C change.
- However:
  - the characteristic equation $\det(sI - A)$ and
  - the input-output representation (transfer function)

  do not change (which is logical, isn't it?).
- Several useful forms, one of them:
  - control canonical form

Control Canonical Form

The system has a transfer function:

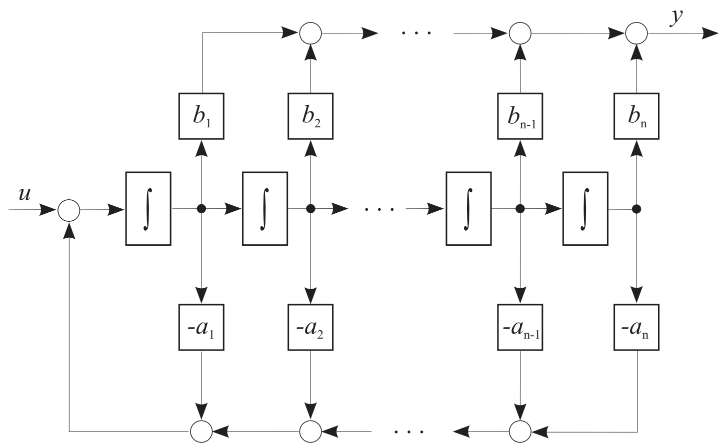
$$G(s) = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

There exists a transformation matrix T such that

$$\dot{z}(t) = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u(t)$$

$$y(t) = (b_1 \quad \dots \quad b_n)\, z(t)$$

Control Canonical Form



Pole Placement in Control Canonical form

$$\dot{z}(t) = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u(t)$$

Coefficients of the characteristic polynomial of $\tilde{A} - \tilde{B}\tilde{K}$:

$$\left(-a_1 - \tilde{k}_1 \quad -a_2 - \tilde{k}_2 \quad \dots \quad -a_n - \tilde{k}_n\right) \quad \Rightarrow \quad \tilde{K}$$

Pole Placement – Ackermann's Formula

1. Transform the model into the control canonical form.
2. Design the controller in this form (which is very easy).
3. Transform the resulting feedback gain vector back.

Ackermann combined these steps into one formula:

$$K = (0 \quad \dots \quad 0 \quad 1)\, W_c^{-1} d(A)$$

where $d(A)$ is the desired characteristic polynomial (in A!), and

$$W_c = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

DC Motor – Position Control

Mathematical Model – Motion Equation

Equation of motion: $J\ddot{y} + b\dot{y} = T_u + v$

v unknown:

Goal: track angle reference, suppress load disturbance

State-Space Model

Equation of motion: $J\ddot{y} + b\dot{y} = k_m u + v$

State-space model:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{b}{J}x_2(t) + \frac{k_m}{J}u(t) + \frac{1}{J}v(t) \end{aligned}$$

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{b}{J} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{k_m}{J} \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ \frac{1}{J} \end{pmatrix} v(t)$$

Controller Design Parameters

$\omega_d = 30\,\mathrm{rad/s}$ (open-loop: $\omega = 9.52$ rad/s)

$\zeta_d = 1$ (two identical real poles in $-\omega_d$)

⇒ settling time of $\approx 0.2\,\mathrm{s}$

Desired characteristic polynomial:

$$d(s) = s^2 + 2\zeta_z\omega_d s + \omega_d^2$$

Asymptotically Constant Disturbances

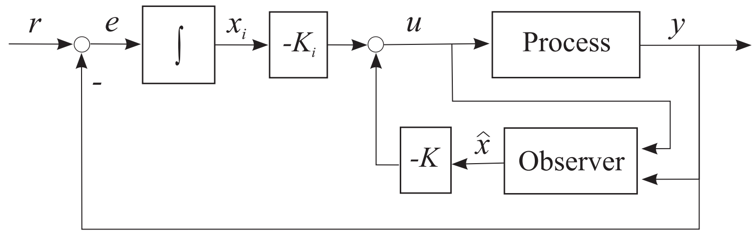
Influence of external forces and friction

Equation of motion: $J\ddot{y} + b\dot{y} = k_m u + v$

v . . . unknown disturbance

In our case, v is mainly due to friction.

Integrator in the Loop



$$\begin{pmatrix} \dot{x}(t) \\ \dot{x}_i(t) \end{pmatrix} = \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x_i(t) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r(t)$$

State-Space: Homework Assignments

- Read Sections 7.1 through 7.6.1.
- Work out examples in this section.
- Work out problems 7.1 – 7.3 (state-space models).
- Work out problems 7.19 – 7.22 (pole placement).

Verify your results by using Matlab.