

Introduction

Solutions to the exam of the course wb2207 taken on January 31, 2007 are given, including a detailed step-by-step explanation of the procedure and references to similar examples in the book, lectures and the assignment. Note that in the actual exam, the detailed explanations are not required and the individual steps in the derivations can often be combined without the loss of clarity.

1 a) Given are the system equations

$$M\ddot{d}(t) = -mg\alpha(t) + u(t) \quad (1)$$

$$L\ddot{\alpha}(t) = \ddot{d}(t) - g\alpha(t) \quad (2)$$

To derive the transfer functions, transform these equations to the Laplace domain

$$s^2MD(s) = -mg\alpha(s) + U(s) \quad (3)$$

$$s^2L\alpha(s) = D(s) - g\alpha(s) \quad (4)$$

Collecting the terms with $\alpha(s)$ in (4), obtain:

$$(Ls^2 + g)\alpha(s) = s^2D(s) \quad (5)$$

and substitute $s^2D(s)$ from the above equation into (3):

$$M(Ls^2 + g)\alpha(s) = -mg\alpha(s) + U(s)$$

The transfer function $G_\alpha(s)$ directly follows:

$$G_\alpha(s) = \frac{\alpha(s)}{U(s)} = \frac{1}{MLs^2 + (M + m)g} \quad (6)$$

Further, express $D(s)$ from (5):

$$D(s) = \frac{Ls^2 + g}{s^2}\alpha(s)$$

and substitute for $\alpha(2)$ from (6):

$$D(s) = \frac{Ls^2 + g}{s^2(MLs^2 + (M + m)g)}U(s)$$

from which the transfer function $G_d(s)$ follows:

$$G_d(s) = \frac{D(s)}{U(s)} = \frac{Ls^2 + g}{s^2(MLs^2 + (M + m)g)}$$

Similar problems: Examples 1 and 2 in Lecture 2, Questions 1a and 1c of the sample exam. Examples 3.17, 3.19, 3.20 and Problems 3.14, 3.15, 3.22, 3.28, 3.29 in the course book.

1 b) Transfer function $G_\alpha(s)$ has a pair of complex poles on the imaginary axis:

$$s_{1,2} = \pm j\sqrt{(M+m)g/ML}$$

and is therefore marginally stable. However, $G_d(s)$ has in addition a double pole in the origin and therefore the overall system is unstable.

Similar problems: See Section 3.7 in the course book and also question 5c of the first Matlab session handout.

1 c) From the definition $x = (d, \alpha, \dot{d}, \dot{\alpha})^T$ we see immediately that $\dot{x}_1(t) = x_3(t)$ and $\dot{x}_2(t) = x_4(t)$. Furthermore, as $\dot{x}_3(t) = \ddot{d}(t)$ and $\dot{x}_4(t) = \ddot{\alpha}(t)$, using equations (1)–(2) we can directly write the four state equations:

$$\begin{aligned}\dot{x}_1(t) &= x_3(t) \\ \dot{x}_2(t) &= x_4(t) \\ \dot{x}_3(t) &= -\frac{mg}{M}x_2(t) + \frac{1}{M}u(t) \\ \dot{x}_4(t) &= -\frac{(M+m)g}{ML}x_2(t) + \frac{1}{ML}u(t)\end{aligned}$$

and express them in the matrix form:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & 0 & 0 \\ 0 & -\frac{(M+m)g}{ML} & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{M} \\ \frac{1}{ML} \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Similar problems: Examples 1 and 2 in Lecture 2, Lecture 7, the Cascaded tanks example of Lecture 12 and Instruction Lecture 2. Question 1d of the sample exam. Examples 7.1 through 7.7 and Problems 7.1 through 7.4 in the course book.

1 d) By writing the transfer function as

$$\frac{Y(s)}{U(s)} = \frac{s^2 + 1}{s^4 + 5s^2}$$

we can write the system in the control canonical form:

$$A = \begin{pmatrix} 0 & -5 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad C = (0 \ 1 \ 0 \ 1) \quad D = (0)$$

Similar problems: Instruction Lecture 2, Lecture 12. Section 7.4 and Problems 7.2, 7.5 in the course book.

2 a) To see whether the system has a finite DC gain, compute first the poles as the roots of $s^2 + 2s + 5$:

$$s_{1,2} = -\frac{2}{2} \pm \frac{\sqrt{4 - 20}}{2} = -1 \pm 2j$$

As the system is stable, we can compute the DC gain by substituting $s = 0$ into the transfer function:

$$\text{DC gain} = G(0) = \frac{1}{5}$$

The system clearly does not have any zeros. By comparing the denominator with the standard form $s^2 + 2\zeta\omega_n s + \omega_n^2$, we obtain $\omega_n = \sqrt{5}$. Finally, by comparing the coefficients at the first power of s , the expression for ζ follows:

$$2\zeta\sqrt{5} = 2 \quad \rightarrow \quad \zeta = \frac{1}{\sqrt{5}}$$

Similar problems: Question 1c of the sample exam. Question 5c of the first Matlab session handout. Question 1.1c of the second Matlab session handout. Various examples in Chapter 3 of the course book.

2 b) Substitute $s = j\omega$ into $G(s)$:

$$G(j\omega) = \frac{1}{5 - \omega^2 + 2j\omega}$$

As the phase of the numerator is 0, the phase of the denominator must be 90° in order to make the overall phase of -90° . The imaginary part of the denominator is positive and therefore the real part must equal zero such that the phase is 90° :

$$5 - \omega^2 = 0 \quad \rightarrow \quad \omega = \sqrt{5} \text{ rad/s}$$

Similar problems: Examples 6.1 and 6.2, Problems 6.1 and 6.2 in the course book.

2 c) Substitute $s = j\omega$ into $G(s)$ and set the magnitude equal to 0.2:

$$|G(j\omega)| = \frac{1}{|5 - \omega^2 + 2j\omega|} = \frac{1}{\sqrt{(5 - \omega^2)^2 + (2\omega)^2}} = \frac{1}{5}$$

Equating the denominators we get:

$$(5 - \omega^2)^2 + (2\omega)^2 = 25$$

which simplifies into:

$$\omega^4 - 6\omega^2 = 0$$

with the solutions $\omega_1 = 0$ rad/s and $\omega_2 = \sqrt{6}$ rad/s. The first solution corresponds to the DC gain and the second one is the frequency at which the magnitude Bode characteristic intersects 0.2, which is just past the resonance peak.

Similar problems: Examples 6.1 and 6.2, Problems 6.1 and 6.2 in the course book.

3 a) Given is the process model

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s}{(s+3)(s^2+9)}$$

the controller

$$C(s) = \frac{U(s)}{E(s)} = K_p + K_d s$$

and the feedback configuration according to Figure 1.

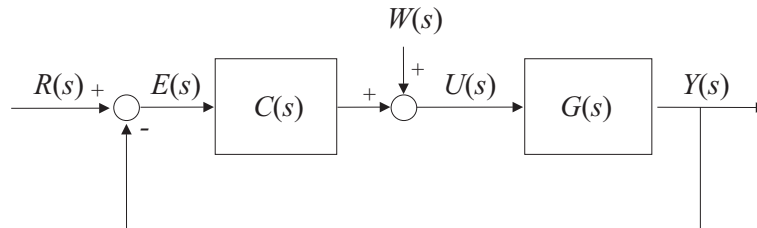


Figure 1: Closed-loop control scheme.

The transfer function $G_w(s)$ from the disturbance $W(s)$ to the error $E(s)$ is given by

$$G_w(s) = \frac{E(s)}{W(s)} = \frac{-G(s)}{1 + G(s)C(s)}$$

This is also easily seen by realizing that $E(s) = R(s) - Y(s) = -Y(s)$, as in our case $R(s) = 0$. Substituting for $G(s)$ en $C(s)$, we have

$$G_w(s) = \frac{\frac{-s}{(s+3)(s^2+9)}}{1 + \frac{K_p s}{(s+3)(s^2+9)}}$$

which simplifies to

$$G_w(s) = \frac{-s}{s^3 + 3s^2 + (9 + K_p)s + 27}$$

Similar problems: Question 2a of the sample exam. Question 1.2 of the second Matlab session handout. Sections 3.2, 4.1, 4.2.3, Examples 4.4, 4.5 and Problems 4.12, 4.17 through 4.21 and 4.32 in the course book.

3 b) Form the closed-loop characteristic equation $1 + G(s)C(s) = 0$

$$1 + \frac{K_p s + K_d s^2}{s^3 + 3s^2 + 9s + 27} = 0$$

By simplifying and collecting the terms at the descending powers of s we get

$$s^3 + (3 + K_d)s^2 + (9 + K_p)s + 27 = 0$$

which can be compared to the desired characteristic equation

$$s^3 + 8s^2 + 24s + 27 = 0$$

Equating the corresponding coefficients, we obtain

$$3 + K_d = 8 \quad \text{and} \quad 9 + K_p = 24$$

which yields $K_p = 15$ and $K_d = 5$.

Similar problems: Question 3a of the sample exam. The disk drive arm control example of Lecture 11. Sections 4.3, 7.5.1, Examples 7.15, 7.17 and Problems 7.19 through 7.23 in the course book.

3 c) The limit is:

$$y_{ss} = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} s G_{cl}(s)R(s) = \lim_{s \rightarrow 0} s \cdot \frac{K_p s + K_d s^2}{s^3 + (3 + K_d)s^2 + (9 + K_p)s + 27} \cdot \frac{1}{s} = 0$$

Similar problems: Examples 1 and 2 in Lecture 4, Question 4d of the sample exam. Question 1.2 of the second Matlab session handout. Section 4.2, Examples 4.1 through 4.5 and Problems 4.5, 4.6 in the course book.

4 The DC gain is 1 (0 dB) and we have the following breakpoints: $\omega = 1$ rad/s (pole), $\omega = 9$ rad/s (zero), $\omega = 300$ rad/s (pole). To sketch the Bode plot, follow the rules of Section 6.1.1.

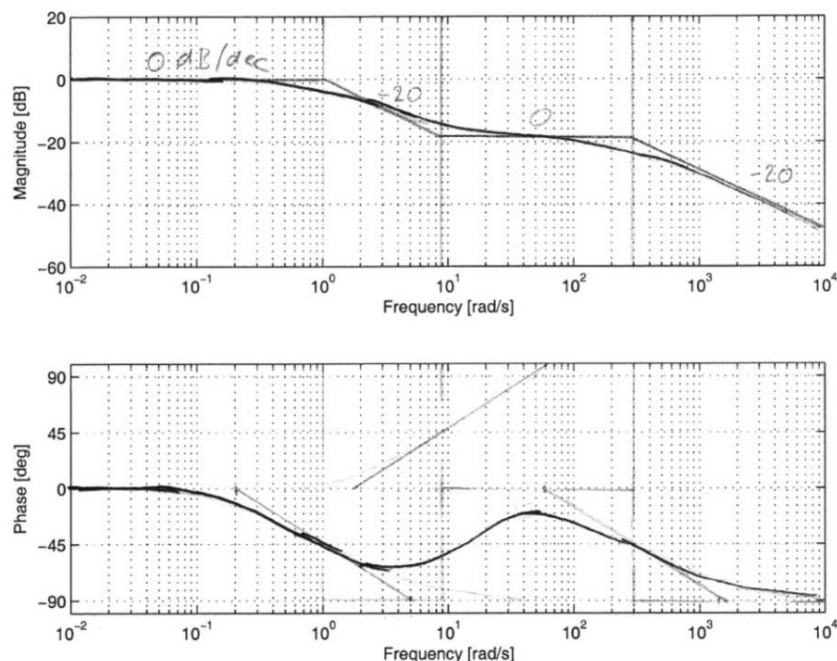


Figure 2: A sketch of the Bode plot.

Similar problems: Bode plot example Lecture 7, DC motor example Lecture 8, Instruction lecture 2, Section 6.1.1, Example 6.3 and Problems 6.3 through 6.8 in the book.

5a) The characteristic equation is

$$1 + K_p \frac{5(s - 1)}{(s + 2)(s + 5)} = 0 \quad (7)$$

and it is already in the required root-locus form $1 + KL(s) = 0$. The transfer function $L(s)$ has two poles in -2 and -5 and one zero in 1. Sketch the root locus, using the rules of Section 5.2.

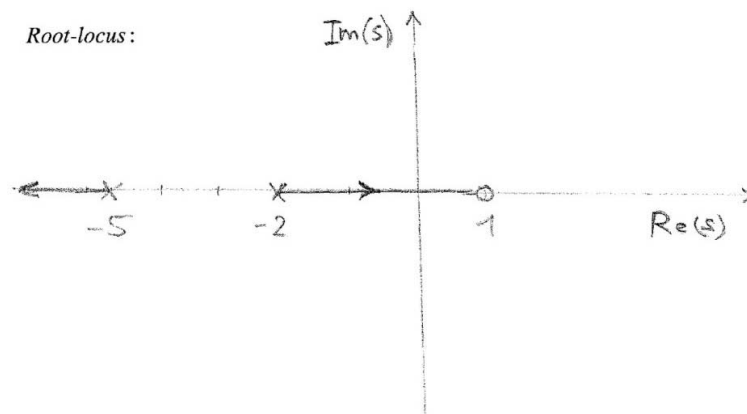


Figure 3: A sketch of the root locus for the characteristic equation (7).

We can clearly see that as K_p becomes larger the root locus passes the imaginary axis to the right half-plane and the closed-loop becomes unstable.

Similar problems: Section 5.2, Example 5.2 and Problems 5.2, 5.3 in the course book.

5b) The gain margin is indicated in Figure 4 and we can see that $GM = 2$ (6 dB). The phase margin is infinite, as the Nyquist plot does not intersect the unit circle.

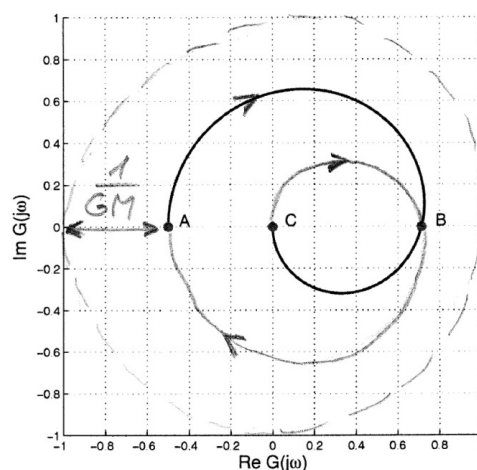


Figure 4: A sketch of the Nyquist plot.

Similar problems: Section 6.3, Examples 6.8, 6.9, 6.10, 6.11 and Problems 6.23, 6.24, 6.25 in the course book.

- 5c) Set $s = j\omega = 0$ in the transfer function to get the DC gain of -0.5 . This corresponds to point A. On the Bode plot, this point cannot be found (zero frequency is not displayed on a logarithmic scale), but the lowest frequency shown (0.001 rad/s) is a good approximation of this situation. Point B corresponds to the phase of 0 degrees and the reasoning about point C is analogous to point A, only for $s = j\omega \rightarrow \infty$

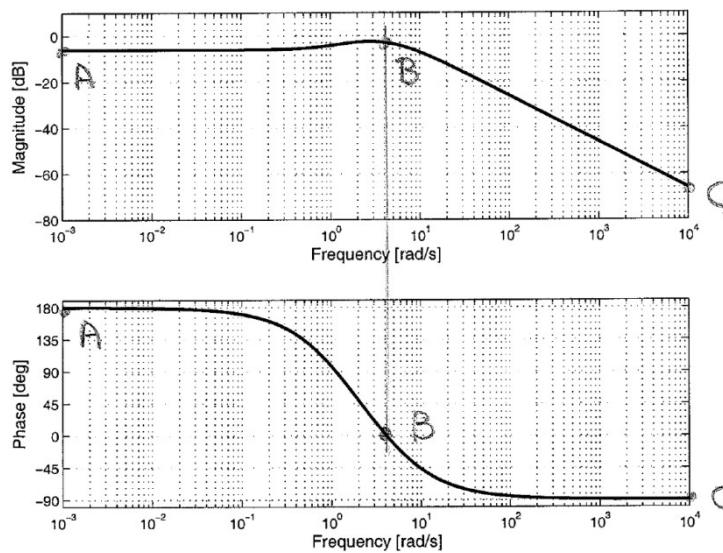


Figure 5: Corresponding points on the Bode plot.

Similar problems: Section 6.3, Example 6.8, 6.9 and Problem 6.19 in the course book.