

## Introduction

Solutions to the exam of the course wb2207 taken on October 31, 2006 are given, including a detailed step-by-step explanation of the procedure and references to similar examples in the book, lectures and the assignment. Note that in the actual exam, the detailed explanations are not required and the individual steps in the derivations can often be combined without the loss of clarity.

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**1 a)** Given are the system equations

$$\ddot{\alpha}(t) = u(t) - k(\alpha(t) - \beta(t)) \quad (1)$$

$$\ddot{\beta}(t) = k(\alpha(t) - \beta(t)) \quad (2)$$

To derive the transfer functions, transform these equations to the Laplace domain

$$s^2\alpha(s) = U(s) - k\alpha(s) + k\beta(s)$$

$$s^2\beta(s) = k\alpha(s) - k\beta(s)$$

and collect the terms with  $\alpha(s)$  and  $\beta(s)$

$$(s^2 + k)\alpha(s) = U(s) + k\beta(s) \quad (3)$$

$$(s^2 + k)\beta(s) = k\alpha(s) \quad (4)$$

To get the transfer function for  $\alpha(s)$ , express  $\beta(s)$  from equation (4)

$$\beta(s) = \frac{k\alpha(s)}{s^2 + k} \quad (5)$$

insert in (3) and multiply the whole equation by  $(s^2 + k)$ :

$$(s^2 + k)^2\alpha(s) = (s^2 + k)U(s) + k^2\alpha(s)$$

Simplifying, we get:

$$(s^4 + 2ks^2)\alpha(s) = (s^2 + k)U(s)$$

and finally:

$$G_\alpha(s) = \frac{\alpha(s)}{U(s)} = \frac{s^2 + k}{s^2(s^2 + 2k)} \quad (6)$$

From (5) and (6), we immediately obtain the transfer function for  $\beta$ :

$$G_\beta(s) = \frac{\beta(s)}{U(s)} = \frac{k}{s^2(s^2 + 2k)}$$

Realizing that  $\alpha_v(s) = s\alpha(s)$  and  $\beta_v(s) = s\beta(s)$ , the transfer functions for the velocities directly follow:

$$G_{\alpha_v}(s) = \frac{\alpha_v(s)}{U(s)} = \frac{s^2 + k}{s(s^2 + 2k)}$$

$$G_{\beta_v}(s) = \frac{\beta_v(s)}{U(s)} = \frac{k}{s(s^2 + 2k)}$$

**Similar problems:** Examples 1 and 2 in Lecture 2, Questions 1a and 1c of the sample exam. Examples 3.17, 3.19, 3.20 and Problems 3.14, 3.15, 3.22, 3.28, 3.29 in the course book.

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**1 b)** This is a good check to see whether we have got at least a part of 1 a) right. We have the initial slope of  $-40$  dB / decade, which means a double pole in zero. This rules out the  $G_{\alpha v}(s)$  and  $G_{\beta v}(s)$  transfer functions. Further we see one resonance peak at  $\omega \approx 7$  rad/s (corresponding to a pair of complex poles) and one anti-resonance peak at  $\omega \approx 5$  rad/s (corresponding to a pair of complex zeros). As  $G_{\beta}(s)$  has no zeros, the only possibility left is  $G_{\alpha}(s)$ . From the numerator of  $G_{\alpha}(s)$  in (6), we have the breakpoint frequency  $\omega = \sqrt{k} = 5$ , which gives  $k = 25$ . Note that we can actually construct the transfer function from the bode plot (knowing that the damping is neglected in the system model).

**Similar problems:** Examples 6.5 through 6.7 and Problems 6.5 through 6.7 in the course book. In Lecture 9, the rotational pendulum example (and demo) was given. This system has a bode plot similar to the one in this question.

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**1 c)** From the definition  $x = (\alpha, \beta, \dot{\alpha}, \dot{\beta})^T$  we see immediately that  $\dot{x}_1(t) = x_3(t)$  and  $\dot{x}_2(t) = x_4(t)$ . Furthermore, as  $\dot{x}_3(t) = \ddot{\alpha}(t)$  and  $\dot{x}_4(t) = \ddot{\beta}(t)$ , using equations (1) and (2) we directly write the four state equations:

$$\begin{aligned}\dot{x}_1(t) &= x_3(t) \\ \dot{x}_2(t) &= x_4(t) \\ \dot{x}_3(t) &= -kx_1(t) + kx_2(t) + u(t) \\ \dot{x}_4(t) &= kx_1(t) - kx_2(t)\end{aligned}$$

and express them in the matrix form:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & -k & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad C = (1 \ 0 \ 0 \ 0) \quad D = 0$$

**Similar problems:** Examples 1 and 2 in Lecture 2, Lecture 7, the Cascaded tanks example of Lecture 12 and Instruction Lecture 2. Question 1d of the sample exam. Examples 7.1 through 7.7 and Problems 7.1 through 7.4 in the course book.

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**1 d)** This system has a double pole in the origin and therefore it is unstable. This can be seen from the magnitude bode plot (initial slope of  $-40$  dB / decade), without actually deriving the transfer function.

**Similar problems:** See Section 3.7 in the course book and also question 5c of the first Matlab session handout.

2 a) Given is the process model

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1}$$

the controller

$$C(s) = \frac{U(s)}{E(s)} = K_p \left( 1 + \frac{1}{T_i s} \right)$$

and the feedback configuration according to Figure 1.

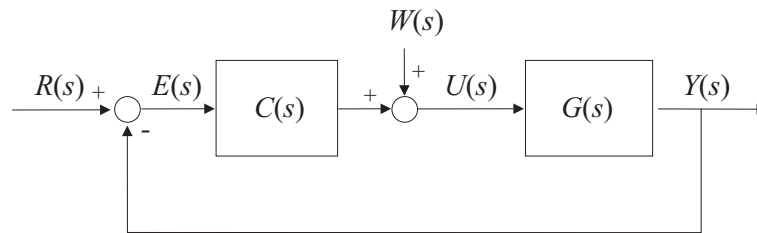


Figure 1: Closed-loop control scheme.

The transfer function  $G_w(s)$  from the disturbance  $W(s)$  to the error  $E(s)$  is given by

$$G_w(s) = \frac{E(s)}{W(s)} = \frac{-G(s)}{1 + G(s)C(s)}$$

This is also easily seen by realizing that  $E(s) = R(s) - Y(s) = -Y(s)$ , as in our case  $R(s) = 0$ . Substituting for  $G(s)$  en  $C(s)$ , we have

$$G_w(s) = \frac{-K}{1 + \frac{K K_p (T_i s + 1)}{T_i s (\tau s + 1)}}$$

which simplifies to

$$G_w(s) = \frac{-K T_i s}{\tau T_i s^2 + T_i (1 + K K_p) s + K K_p}$$

**Similar problems:** Question 2a of the sample exam. Question 1.2 of the second Matlab session handout. Sections 3.2, 4.1, 4.2.3, Examples 4.4, 4.5 and Problems 4.12, 4.17 through 4.21 and 4.32 in the course book.

2 b) The limit is:

$$e_{ss} = \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} s G_w(s) W(s) = \lim_{s \rightarrow 0} s \cdot \frac{-K T_i s}{\tau T_i s^2 + T_i (1 + K K_p) s + K K_p} \cdot \frac{3}{s} = 0$$

**Similar problems:** Examples 1 and 2 in Lecture 4, Question 4d of the sample exam. Question 1.2 of the second Matlab session handout. Section 4.2, Examples 4.1 through 4.5 and Problems 4.5, 4.6 in the course book.

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2 c) Form the closed-loop characteristic equation  $1 + G(s)C(s) = 0$

$$1 + \frac{KK_p}{\tau s + 1} \frac{T_i s + 1}{T_i s} = 0$$

Simplifying and collecting the terms at the descending powers of  $s$  we get

$$s^2 + \frac{1 + KK_p}{\tau} s + \frac{KK_p}{\tau T_i} = 0$$

which can be compared to the desired characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Equating the corresponding coefficients, we get

$$2\zeta\omega_n = \frac{1 + KK_p}{\tau} \quad \text{and} \quad \omega_n^2 = \frac{KK_p}{\tau T_i}$$

which yields

$$K_p = \frac{2\zeta\omega_n\tau - 1}{K} \quad \text{and} \quad T_i = \frac{KK_p}{\tau\omega_n^2} = \frac{2\zeta\omega_n\tau - 1}{\tau\omega_n^2}$$

**Similar problems:** Question 3a of the sample exam. The disk drive arm control example of Lecture 11. Sections 4.3, 7.5.1, Examples 7.15, 7.17 and Problems 7.19 through 7.23 in the course book.

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3 a) There are essentially two methods to show that the state-space model

$$\dot{x}(t) = \begin{pmatrix} -3 & 3 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 3 \end{pmatrix} u(t) \quad (7)$$

$$y(t) = (1 \ 1) x(t) \quad (8)$$

corresponds to the transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{3(s+6)}{(s+3)(s+2)}$$

**Method 1:** use the formula

$$G(s) = C(sI - A)^{-1}B + D,$$

substitute for  $A$ ,  $B$ ,  $C$  and  $D$  from equations (7) and (8)

$$G(s) = (1 \ 1) \left[ \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -3 & 3 \\ 0 & -2 \end{pmatrix} \right]^{-1} \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

and work out the matrix inverse

$$G(s) = \frac{1}{(s+3)(s+2)} (1 \ 1) \text{adj} \begin{pmatrix} s+3 & -3 \\ 0 & s+2 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Computing the adjugate matrix in this two-by-two case means swapping the diagonal elements and inverting the sign at the off-diagonal elements. Working out the matrix product yields

$$\frac{Y(s)}{U(s)} = \frac{3s + 18}{(s + 3)(s + 2)}$$

**Method 2:** write the state-space model as a set of differential equations

$$\begin{aligned} \dot{x}_1(t) &= -3x_1(t) + 3x_2(t) \\ \dot{x}_2(t) &= -2x_2(t) + 3u(t) \end{aligned}$$

Transform these equations to the Laplace domain and collect the terms

$$\begin{aligned} (s + 3)X_1(s) &= 3X_2(s) \\ (s + 2)X_2(s) &= 3U(s) \end{aligned}$$

Express  $X_2(s)/U(s)$  from the second equation

$$\frac{X_2(s)}{U(s)} = \frac{3}{s + 2}$$

and substitute to the first one to get

$$\frac{X_1(s)}{U(s)} = \frac{9}{(s + 3)(s + 2)}$$

From the output equation (8), we see that  $Y(s) = X_1(s) + X_2(s)$ , and therefore

$$\frac{Y(s)}{U(s)} = \frac{9}{(s + 3)(s + 2)} + \frac{3}{s + 2} = \frac{3s + 18}{(s + 3)(s + 2)}$$

**Similar problems:** Question 3b of the sample exam. Lecture 11. Section 7.4.2, Example 7.12 and Problem 7.14 in the course book.

**3b)** As the transfer function is given

$$\frac{Y(s)}{U(s)} = \frac{3s + 18}{s^2 + 5s + 6}$$

writing the system in control canonical form is straightforward:

$$A = \begin{pmatrix} -5 & -6 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = ( 3 \quad 18 ) \quad D = 0$$

**Similar problems:** Instruction Lecture 2, Lecture 12. Section 7.4 and Problems 7.2, 7.5 in the course book.

3 c) Use the formula

$$K_{ff} = \frac{d(0)}{b(0)} \cdot DC_{des}$$

As all the elements are given:  $d(s) = s^2 + 15s + 50$ ,  $b(s) = 3(s + 6)$  and  $DC_{des} = 9$ , we can substitute  $s = 0$  (realizing that  $d(s)$  has two negative roots and that the closed loop is therefore stable)

$$K_{ff} = \frac{50}{18} \cdot 9 = 25$$

**Similar problems:** Question 3b and 3c of the sample exam. Lecture 11.

4 The DC gain is 100 and we have the following breakpoints:  $\omega = 0.2 \text{ rad/s}$  (pole),  $\omega = 2 \text{ rad/s}$  (zero),  $\omega = 10 \text{ rad/s}$  (zero),  $\omega = 50 \text{ rad/s}$  (pole). To sketch the Bode plot, follow the rules of Section 6.1.1.

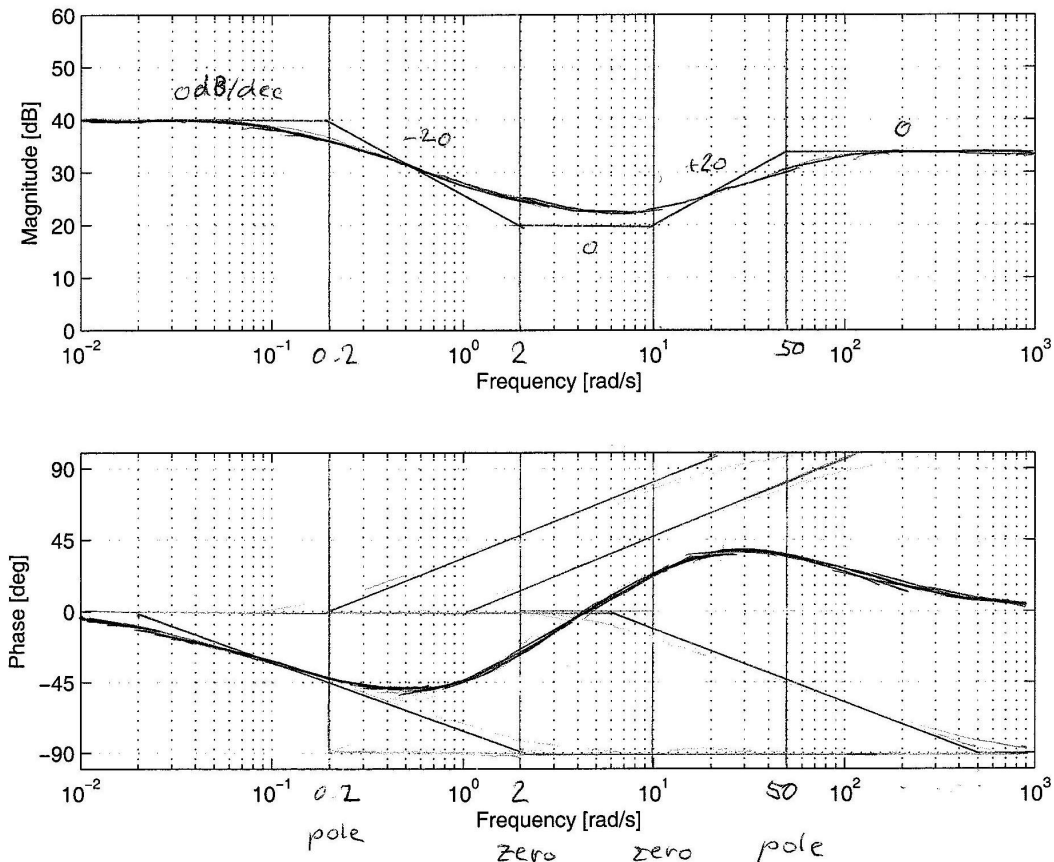


Figure 2: A sketch of the Bode plot.

**Similar problems:** Bode plot example Lecture 7, DC motor example Lecture 8, Instruction lecture 2, Section 6.1.1, Example 6.3 and Problems 6.3 through 6.8 in the course book.

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**5a)** This double integrator has the phase of  $-180^\circ$  for all  $\omega$ . The phase margin (PM) is therefore zero, regardless of the cross-over frequency. To achieve  $\text{PM} = 45^\circ$ , the controller has to add exactly these  $45^\circ$  at the frequency  $\omega_c$ . The phase of  $D(j\omega_c)$  is  $\tan^{-1}(T_d\omega_c)$ , from which we compute

$$T_d = \frac{\tan 45^\circ}{\omega_c} = \frac{1}{0.1} = 10$$

The magnitude  $|G(j\omega_c)D(j\omega_c)|$  is

$$|G(j\omega_c)D(j\omega_c)| = K \left| \frac{j\omega_c T_d + 1}{(j\omega_c)^2} \right| = K \left| \frac{j + 1}{-0.1^2} \right| = 100\sqrt{2}K$$

At the cross-over frequency the magnitude must be 1, which gives

$$K = \frac{1}{100\sqrt{2}} = 0.00707$$

**Similar problems:** This is an exact blueprint of assignment 2.3 of the second Matlab session handout. The same problem (satellite control) was worked out in Lecture 7. See also Section 6.7 in the course book.

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**5b)** The characteristic equation is

$$1 + \frac{1}{s^2}(T_d s + 1) = 0$$

To get this equation in the required form, multiply by  $s^2$  and collect the terms without  $T_d$  and with  $T_d$

$$\underbrace{s^2 + 1}_{a(s)} + T_d \underbrace{s}_{b(s)} = 0 \quad (9)$$

Dividing by  $s^2 + 1$  yields

$$1 + T_d \frac{s}{s^2 + 1} = 0 \quad (10)$$

and therefore

$$L(s) = \frac{s}{s^2 + 1}$$

**Similar problems:** A similar problem (varying load inertia of a DC motor) was worked out in Lecture 6. See also Examples 5.2, 5.14 and Problems 5.1, 5.17, 5.18 in the course book.

- 5c) The transfer function  $L(s)$  has a pair of complex poles in  $+j$  and  $-j$  and one zero in 0. Sketch the root locus, using the rules of Section 5.2.

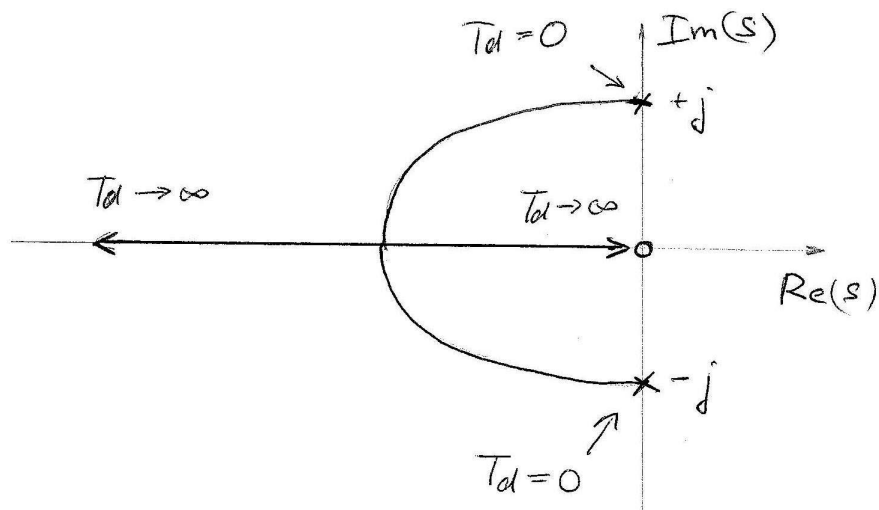


Figure 3: A sketch of the root locus for the characteristic equation (10).

In this case, the exact location of the ‘breakin’ point can easily be computed (for  $T_d = 2$ , equation (9) has a double real pole in  $-1$ ), but for a rough sketch this is not necessary.

**Similar problems:** Section 5.2, Example 5.2 and Problems 5.2, 5.3 in the course book.