I. PRINCIPLES OF QUANTUM MECHANICS

(I.1) (a) Calculate the de Broglie wavelength for an electron with kinetic energy of 1 eV. Kinetic energy is expressed by

\[ E = \frac{1}{2}mv^2. \]  

(1)

It can be expressed by using momentum \( p = mv \) as follows

\[ E = \frac{p^2}{2m}. \]  

(2)

Substituting the equation in terms of the momentum into de Broglie wavelength equation: \( \lambda = \frac{h}{p} \) gives

\[ \lambda = \frac{h}{\sqrt{2Em}}. \]  

(3)

Substituting physical constants and kinetic energy into the equation 3 gives the following de Broglie wavelength:

\[ 1.23 \times 10^{-9} \text{ m} \]

(I.2) An electron and a photon have the same energy. At what value of energy (in eV) will the wavelength of the photon be 10 times that of the electron?

Solution:

Kinetic energy \( E_e \) of an electron can be expressed using the de Broglie wavelength \( \lambda e \) as

\[ E_e = \frac{(h/\lambda e)^2}{2m}. \]  

(4)

Energy of a photon \( E_p \) can be expressed by using the wavelength \( \lambda p \)

\[ E_p = h \nu = h \frac{c}{\lambda p}. \]  

(5)

The ratio of these wavelength was given by the question as follows

\[ \lambda e = \frac{\lambda p}{10}. \]  

(6)

Using the condition \( E_e = E_p \) and substitution of the equation 6 result

\[ h \frac{c}{\lambda p} = \frac{100h^2}{2m\lambda p^2}; \]  

(7)

hence

\[ \lambda p = \frac{100h}{2mc}. \]  

(8)

Substituting this into equation 5 makes the energy for the both

\[ E = \frac{mc^2}{50}. \]  

(9)
Using the constants for electron mass $m$ and light velocity $c$, we obtain $E = 1.6 \times 10^{-15}$ J.

**Solution:**

The uncertainty in position is 12 Å for a particle of mass $5 \times 10^{-29}$ kg. Determine the minimum uncertainty in the momentum of the particle.

**Solution:**

The uncertainty in the momentum $\Delta p$ can be obtained by using $\Delta x \times \Delta p \geq \hbar$. $\Delta x$ is given, thus:

$$\Delta p = \frac{1.054 \times 10^{-34}}{12 \times 10^{-10} \text{ kg} \times \text{m/s}} = 8.78 \times 10^{-26} \text{ kg m/s}$$

### II. Wave

**Solution:**

- **(II.1)** $A \cos \frac{2\pi}{\lambda} x$
- **(II.2)** $A \cos \frac{2\pi}{T} t$
- **(II.3)** $A \cos (\frac{2\pi}{T} t - \frac{2\pi}{\lambda} x)$
- **(II.4)** $A \left( \exp^{j\left(\frac{2\pi}{T} t - \frac{2\pi}{\lambda} x\right)} + \exp^{-j\left(\frac{2\pi}{T} t - \frac{2\pi}{\lambda} x\right)} \right)$
- **(II.5)** The phase velocity $v = \frac{\lambda}{T} = \frac{\omega}{k}$

### III. General Solution of Wave Function

**Solution:**

- **(III.1)** Applying Euler’s formula to the general solution (2) gives
  
  $$\psi(x) = A \cos (|c|x) + j \sin (|c|x) + B \cos (|c|x) - j \sin (|c|x) = (A + B) \cos (|c|x) + j(A - B) \sin (|c|x)$$

  Setting new constants:

  $$A + B \rightarrow C, \quad j(A - B) \rightarrow D$$

  $$\psi(x) = C \cos (|c|x) + D \sin (|c|x)$$

  **(12)**

- **(III.2)**
  
  (a) If $E > V_0$,

  $$\psi(x) = Ae^{jKx} + Be^{-jKx}$$

  or

  $$\psi(x) = A \cos (Kx) + B \sin (Kx)$$

  where

  $$K = \sqrt{\frac{2m}{\hbar^2} (E - V_0)}$$

  **(15)**

  (b) If $E < V_0$,

  $$\psi(x) = Ae^{Kx} + Be^{-Kx}$$

  where

  $$K = \sqrt{\frac{2m}{\hbar^2} (V_0 - E)}$$

  **(17)**

- **(III.3)** $\Psi(x, t) = \psi(x)\phi(t) = A \exp^{jKx} \times \exp^{-j\frac{\omega}{\hbar}t} = A \exp^{j(kx - \frac{\omega}{\hbar}t)}$, where $k > 0$, therefore the wave travels to the positive direction.
IV. Applications of Schrödinger’s wave equation

(IV.1) An electron in free space is described by a plane wave given by \( \Psi(x, t) = Ae^{i(kx - \omega t)} \) where \( k = 1.5 \times 10^9 \text{m}^{-1} \) and \( \omega = 1.5 \times 10^{13} \text{ rad/s} \).

(a) Determine the phase velocity of the plane wave.

(b) Calculate the wavelength, momentum and kinetic energy (in eV) of the electron.

Solution:

(a) \( kx - \omega t = \) constant, then

\[
\frac{k}{\partial t} - \omega = 0 \Rightarrow \frac{\partial x}{\partial t} = v_p = \frac{\omega}{k}
\]

\[
v_p = \frac{1.5 \times 10^{13}}{1.5 \times 10^9} = 10^4 \text{[m/s]} = 10^6 \text{[cm/s]}
\]

(b)

\[
k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi}{1.5 \times 10^9} = 41.9 \text{Å}
\]

\[
p = \frac{h}{\lambda} = \frac{6.625 \times 10^{-34}}{41.9 \times 10^{-10}} = 1.58 \times 10^{-25} \text{[kg \cdot m/s]}
\]

\[
E = \frac{1}{2}mv^2 = \frac{1}{2} \frac{p^2}{m} = \frac{1}{2} \left( \frac{1.58 \times 10^{-25}}{9.1 \times 10^{-31}} \right)^2 = 1.37 \times 10^{-20} \text{J}(8.56 \times 10^{-2} \text{eV})
\]