Exercise Solid-State Physics (ET2908 and ET3175-D1) 2008-2009Q1: dr. R. Ishihara, DIMES-TC0.0044, r.ishihara@tudelft.nl

Solution for Exercise-sheet 2.1a

I. Principles of Quantum Mechanics

(I.1) (a) Calculate the de Broglie wavelength for an electron with kinetic energy of 1 eV.
Kinetic energy is expressed by

$$E = \frac{1}{2}mv^2. \tag{1}$$

It can be expressed by using momentum $p = mv$ as follows

$$E = \frac{p^2}{2m}. \tag{2}$$

Substituting the equation in terms of the momentum into de Broglie wavelength equation: $\lambda = \frac{h}{p}$ gives

$$\lambda = \frac{h}{\sqrt{2Em}}. \tag{3}$$

Substituting physical constants and kinetic energy into the equation 3 gives the following de Broglie wavelength:
1.23×10^{-9} m

(I.2) An electron and a photon have the same energy. At what value of energy (in eV) will the wavelength of the photon be 10 times that of the electron?
Solution:
Kinetic energy E_e of an electron can be expressed using the de Broglie wavelength λe as

$$E_e = \frac{(h/\lambda e)^2}{2m}. \tag{4}$$

Energy of a photon E_p can be expressed by using the wavelength λp

$$E_p = h\nu = h\frac{c}{\lambda p}. \tag{5}$$

The ratio of these wavelength was given by the question as follows

$$\lambda e = \frac{\lambda p}{10}. \tag{6}$$

Using the condition $E_e = E_p$ and substitution of the equation 6 result

$$h\frac{c}{\lambda p} = \frac{100h^2}{2m\lambda p^2}, \tag{7}$$

hence

$$\lambda p = \frac{100h}{2mc}. \tag{8}$$

Substituting this into equation 5 makes the energy for the both

$$E = \frac{mc^2}{50}. \tag{9}$$

Using the constants for electron mass m and light velocity c, we obtain $\underline{E = 1.6 \times 10^{-15} \text{ J}}$.

(I.3) The uncertainty in position is 12 Å for a particle of mass 5×10^{-29}kg. Determine the minimum uncertainty in the momentum of the particle.

Solution:

The uncertainty in the momentum Δp can be obtained by using $\Delta x \times \Delta p \geq \hbar$. Δx is given, thus: $\Delta p = \frac{1.054 \times 10^{-34}}{12 \times 10^{-10}} = \underline{8.78 \times 10^{-26} \text{kg} \times \text{m/s}}$

II. Wave

(II.1) Solution: $A \cos \frac{2\pi}{\lambda} x$

(II.2) Solution: $A \cos \frac{2\pi}{T} t$

(II.3) Solution: $A \cos \left(\frac{2\pi}{T} t - \frac{2\pi}{\lambda} x \right)$

(II.4) Solution: $\frac{A}{2} \left(\exp^{j\left(\frac{2\pi}{T} t - \frac{2\pi}{\lambda} x\right)} + \exp^{-j\left(\frac{2\pi}{T} t - \frac{2\pi}{\lambda} x\right)} \right)$

(II.5) Solution: The phase velocity $v = \frac{\lambda}{T} = \frac{\omega}{k}$

III. General Solution of Wave Function

(III.1)

Applying Euler's formula to the general solution (2) gives

$$\psi(x) = A\left[\cos\left(|c|x\right) + j \sin\left(|c|x\right)\right] + B\left[\cos\left(|c|x\right) - j \sin\left(|c|x\right)\right] = (A+B)\cos\left(|c|x\right) + j(A-B)\sin\left(|c|x\right) \tag{10}$$

Setting new constants:

$$A + B \rightarrow C \quad , \qquad j(A-B) \rightarrow D \tag{11}$$

$$\psi(x) = C \cos\left(|c|x\right) + D \sin\left(|c|x\right) \tag{12}$$

(III.2)

(a) If $E > V_0$,

$$\psi(x) = Ae^{jKx} + Be^{-jKx} \tag{13}$$

or

$$\psi(x) = A \cos(Kx) + B \sin(Kx) \tag{14}$$

where

$$K = \sqrt{\frac{2m}{\hbar^2}(E - V_0)} \tag{15}$$

(b) If $E < V_0$,

$$\psi(x) = Ae^{Kx} + Be^{-Kx} \tag{16}$$

where

$$K = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} \tag{17}$$

(III.3) $\Psi(x,t) = \psi(x)\phi(t) = A \exp^{jkx} \times \exp^{-j\frac{E}{\hbar}t} = A \exp^{j\left(kx - \frac{E}{\hbar}t\right)}$, where $k > 0$, therefore the wave travels to the positive direction.

(IV.1) An electron in free space is described by a plane wave given by $\Psi(x,t) = Ae^{j(kx-\omega t)}$ where $k = 1.5 \times 10^9 \mathrm{m}^{-1}$ and $\omega = 1.5 \times 10^{13}$ rad/s.

(a) Determine the phase velocity of the plane wave.

(b) Calculate the wavelength, momentum and kinetic energy (in eV) of the electron.

Solution:

(a) $kx - \omega t = \text{constant}$, then

$$k\frac{\partial x}{\partial t} - \omega = 0 \rightarrow \frac{\partial x}{\partial t} = v_p = \frac{\omega}{k} \tag{18}$$

$$v_p = \frac{1.5 \times 10^{13}}{1.5 \times 10^9} = 10^4[\mathrm{m/s}] = \underline{10^6[\mathrm{cm/s}]} \tag{19}$$

(b)

$$k = \frac{2\pi}{\lambda} \rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi}{1.5 \times 10^9} = \underline{41.9\text{Å}} \tag{20}$$

$$p = \frac{h}{\lambda} = \frac{6.625 \times 10^{-34}}{41.9 \times 10^{-10}} = \underline{1.58 \times 10^{-25}[\mathrm{kg \cdot m/s}]} \tag{21}$$

$$E = \frac{1}{2}mv^2 = \frac{1}{2}\frac{p^2}{m} = \frac{1}{2}\frac{\left(1.58 \times 10^{-25}\right)^2}{9.1 \times 10^{-31}} = \underline{1.37 \times 10^{-20}\mathrm{J}(8.56 \times 10^{-2}\mathrm{eV})}. \tag{22}$$