Solution for Exercise-sheet 4

I. The potential barrier

(I.1) Estimate the tunneling probability of an electron tunneling through a rectangular barrier with a barrier height V_0 of 1eV and a barrier width of 15Å. The electron energy is 0.20 eV.

Solution:

$K_2 = \sqrt{\frac{2m(V_0-E)}{\hbar^2}} = \left(\frac{2(9.11\times10^{-31})(1-0.2)(1.6\times10^{-19})}{(1.054\times10^{-34})^2}\right)^{\frac{1}{2}} = 4.58\times10^9\text{m}^{-1}$

$T \cong 16\left(\frac{0.2}{1}\right)\left(1-\frac{0.2}{1}\right)\left[-2(4.58\times10^9)(15\times10^{-10})\right] = 2.76\times10^{-6}$

II. The hydrogen atoms

Time-independent Schrödinger equation for the 2D hydrogen atom

(II.1)

Solution:

In cylindrical coordinates the Schrödinger equation for the 2D hydrogen atom is written as

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi(r,\theta)-\frac{e^2}{4\pi\epsilon_0 r}\psi(r,\theta)=E\psi(r,\theta).$$

(II.2)

Solution:

If ψ does not depend on θ, i.e. $\psi=\psi(r)$, then $\partial\psi/\partial\theta=0$. Substitution in the above equation gives

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi(r)}{\partial r^2}+\frac{1}{r}\frac{\partial\psi(r)}{\partial r}\right]-\frac{e^2}{4\pi\epsilon_0 r}\psi(r)=E\psi(r).$$

(II.3)

Solution:

We next try $\psi(r)=e^{-\eta r}$ as a solution. The result is

$$-\frac{\hbar^2}{2m}\left[\eta^2e^{-\eta r}-\frac{\eta}{r}e^{-\eta r}\right]-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}e^{-\eta r}=Ee^{-\eta r},$$

or

$$\left(-\frac{\hbar^2\eta^2}{2m}-E\right)e^{-\eta r}+\left(\frac{\hbar^2\eta}{2m}-\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r}e^{-\eta r}=0.$$

Since this relationship should be valid for every r, the both terms in the two brackets should vanish, so that;

$$E=-\frac{\hbar^2\eta^2}{2m},\ \eta=\frac{2me^2}{4\pi\epsilon_0\hbar^2}.$$

(II.4)

Solution:

From previous equations one finds that

$$E = -\frac{me^4}{8\pi^2\epsilon_0^2\hbar^2} = \underline{-54.4\text{eV}}.$$

The hydrogen atom

(II.5) (Question 2.39 of the Neaman book Chapter 2 page 55)

Solution: For the $1s$ state ($n = 1, l = 0$ and $m = 0$), the wave function is given by

$$\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2}\exp\left(\frac{-r}{a_0}\right)$$

(page 47 equation 2.73), where the a_0 is the Bohr radius and hence the RADIAL probability $P(r)$

$$P(r) = 4\pi r^2\psi_{100}\psi_{100}^* = 4\pi r^2\frac{1}{\pi}\left(\frac{1}{a_0}\right)^3\exp\left(\frac{-2r}{a_0}\right)$$

or

$$P(r) = \frac{4}{(a_0)^3}r^2\exp\frac{-2r}{a_0}.$$

Note that the differential volume of the shell around the nucleus is multiplied to the product of $\psi_{100}\cdot\psi_{100}^*$. To find the maximum probability

$$\frac{dP(r)}{dr} = 0$$

$$= \frac{4}{(a_0)^3}\left[r^2\left(\frac{-2}{a_0}\right)\exp\left(\frac{-2r}{a_0}\right) + 2r\exp\left(\frac{-2r}{a_0}\right)\right]$$

which gives

$$0 = \frac{-r}{a_0} + 1$$

$$\underline{r = a_0}$$

<u>$r = a_0$ is the radius that gives the greatest probability.</u>